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Branching Ratios for the Decay of p-Shell Λ-Hypernuclei (*).

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Summary. — The possibility of determining the spin dependence of the Λ-nucleon force from a study of hypernuclei in which the Λ particle is bound to a nucleus with an incompletely filled ($1p$) nuclear shell is investigated. It is pointed out that present uncertainties in the data on binding energies of hypernuclei permit only the determination of the combination E_s+3E_t, where E_s and E_t are the singlet and triplet interaction energies, respectively. It is shown that the decay of hypernuclei by π^- emission can give information regarding the spin dependence of the force if the branching ratio for decay to the ground state and first excited state of the daughter nucleus is measured. In particular, if the spin of $^{12}B_\Lambda$ is 2, decay to the first excited state of ^{12}C is favored by a factor 1.2 over decay to the ground level. However, if $J=1$ for this hypernucleus, decay to the ground state is preferred by a factor of 2.4. The change that the interaction of the π^--meson with the ^{12}C nucleus produces in the branching ratio is estimated by use of a complex square well potential. Inclusion of the interaction significantly increases the branching ratio for a hypernuclear spin of 2, but does not change the ratio for $J=1$. On the other hand, for $^7Li_\Lambda$ the branching ratio for decay to the ground and first excited state of ^7Be is independent of the spin of hypernucleus.

(*) Work done partly under the auspices of the U. S. Atomic Energy Commission.

1. – Binding energies.

The binding energies of hypernuclei arise from interactions of the Λ-particle with the contituent nucleons of the nucleus. From the data [1] on $^5He_\Lambda$, one can conclude that the binding of the Λ to the s-state nucleons is 3 MeV. The remainder is due to interaction with the p-shell nucleons and must be a function of E_t and E_s, the triplet and singlet interaction energies. Thus the total binding energy is

$$\text{B. E. (MeV)} = 3 + \alpha E_t + \beta E_s \,,$$

where $\alpha+\beta = N$, the number of p-state particles. We have used the convention that attractive forces give rise to positive energies. The assumption has been made that the ^{4}He core is not altered by the addition of the p-state nucleons.

For either j-j or L-S coupling, α and β are easily found in the following way. We assume the interaction energy between a Λ-particle and a nucleon may be written as

$$V = V_s(1 - \sigma\cdot\sigma_\Lambda)/4 + V_t(3 + \sigma\cdot\sigma_\Lambda)/4 \,, \tag{1}$$

where V_s and V_t are the singlet and triplet potentials, respectively, and the spin functions multiplying them are the appropriate projection operators. This can be rewritten as

$$V = \tfrac{1}{4}(V_s + 3V_t) + \tfrac{1}{4}(V_t - V_s)\sigma\cdot\sigma_\Lambda \,. \tag{2}$$

We shall now use equation (2) to calculate the binding due to the p-shell nucleons.

a) j-j coupling. – Define the matrix elements of V_s and V_t as E_s and E_t, respectively (*). To take matrix elements of $\sigma\cdot\sigma_\Lambda$ we notice that

$$\langle\sigma\cdot\sigma_\Lambda\rangle = \frac{(\sigma\cdot j)\langle\sigma_\Lambda\cdot j\rangle}{j(j+1)} \,, \tag{3}$$

and when $j = \frac{3}{2}$ or $j = \frac{1}{2}$

$$\langle\sigma\cdot\sigma_\Lambda\rangle = \pm\tfrac{2}{3}\langle\sigma_\Lambda\cdot j\rangle \,, \tag{4}$$

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[1] R. Ammar, R. Levi Setti, W. Slater, S. Limentani, P. E. Schlein and P. H. Steinberg: *Nuovo Cimento*, **15**, 181 (1960).

(*) Since the Λ particle is assumed bound in an s-state only a radial integration is involved (*i.e.*, only $k=0$ in the Slater decomposition of the potential is important).

respectively. Thus we can write the interaction energy between a particle (bound in an s-state) and the i-th nucleon as

$$E_i = a + b\,\langle j_i \cdot j_\Lambda \rangle\,, \tag{5}$$

where

$$\begin{cases} a = \frac{1}{4}(E_s + 3E_t)\,, \\ b = \pm\,\frac{1}{3}(E_t - E_s)\,. \end{cases} \tag{6}$$

Here $\langle j_i \cdot j_\Lambda \rangle$ means the matrix element of $j_i \cdot j_\Lambda$ and we have made use of the fact that $j_\Lambda = \frac{1}{2}\sigma_\Lambda$.

To find the total binding energy we must sum equation (5) over all the p state nucleons. If it is assumed that the nuclear configuration is pure, then $\sum b j_i = b \sum j_i = b J_N$. Thus

$$E = Na + b\langle J_N \cdot j_\Lambda \rangle\,, \tag{7}$$

where N is the number of p nucleons. The expectation value of $J_N \cdot j_\Lambda$ is simple to find so that ultimately we obtain, for the total binding energy including the contribution from the s nucleons (in MeV), the expression

$$\text{B.E.} = 3 + Na + \frac{b}{2}\left[J(J+1) - J_N(J_N+1) - \frac{3}{4}\right], \tag{8}$$

where J is the spin of the hyperfragment ([2]).

A plot of the binding energy data, Fig. 1, shows that because of the large

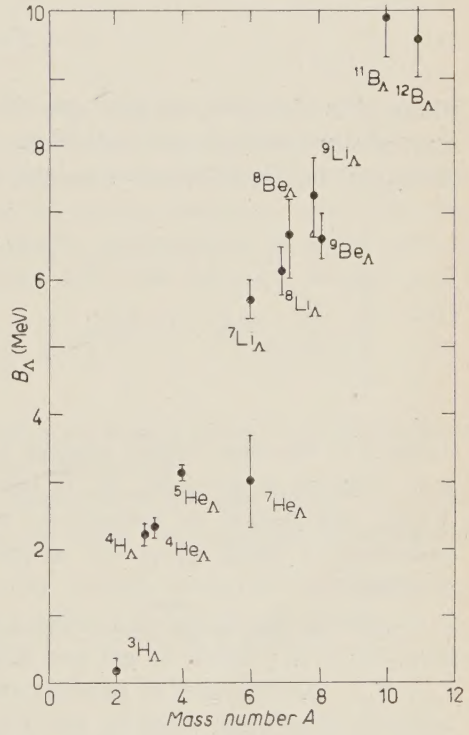

Fig. 1. – The experimental binding energies of a Λ-particle to nuclei of mass number A. Isotopes are displaced slightly to increase legibility. The data are taken from ref. ([1]).

([2]) Eq. (8) shows that, provided the size of the nucleus and the coupling scheme (in this case j-j) does not change appreciably from nucleus to nucleus, there exist geometrical relationships between the binding energies of various hypernuclei. Such geometrical relationships have been exploited both in atomic spectroscopy R. F. BACHER and S. GOUDSMIT: *Phys. Rev.*, **46**, 948 (1934)] and in nuclear spectroscopy [I. TALMI. and R. THIEBERGER: *Phys. Rev.*, **103**, 718 (1956)]. We should like to thank Dr. A. K. KERMAN for pointing out to us how simply such relationships may be derived.

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experimental uncertainties, the data can be fitted by a straight line, so only $a = \frac{1}{4}(E_s + 3E_t)$ is determined.

b) L-S coupling. – We return to equation (2) and note that in *L-S* coupling the spin of the individual nucleons is a good quantum number. We can then write

$$E = a + b\langle s_i \cdot j_\Lambda \rangle \,, \tag{9}$$

where a has the value given by equation (6) and

$$b = E_t - E_s \,. \tag{10}$$

After summing over all the p-state nucleons it is found that the interaction energy is

$$E = Na + b\langle S \cdot j_\Lambda \rangle \,, \tag{11}$$

where S is the total nuclear spin ($S = \sum s_i$). In this case the calculation is again straightforward and leads to matrix elements of the Λ-nucleon interaction including the contribution from the s-state particles, given by

$$\begin{cases} \langle H \rangle = (3 + Na)\,\delta(J_N, \bar{J}_N) + b/2(-1)^{J_N+\bar{J}_N}[(2J_N+1)((2\bar{J}_N+1)]^{\frac{1}{2}} \\ \quad \sum_{J_3=|S-\frac{1}{2}|}^{S+\frac{1}{2}} (2J_3+1)\,[J_3(J_3+1) - S(S+1) - 3/4] \cdot \\ \qquad \cdot W(SJ_3J_NJ;\, \tfrac{1}{2}L)W(SJ_3\bar{J}_NJ;\, \tfrac{1}{2}L) \,, \end{cases} \tag{12}$$

where L is the total orbital angular momentum of the nucleons, S is the total spin angular momentum, J_3 is the result of combining nuclear spin and Λ spin and hence is limited to values of $S \pm \frac{1}{2}$, and W is the Racah coefficient [3] which gives the amplitude of the spin state J_3 in the wave function.

Equation (12) shows that when *L-S* coupling is used for the nuclear wave functions it is possible to get non diagonal matrix elements of the interaction ($J_N \neq \bar{J}_N$). An example is provided by $^8\mathrm{Li}_\Lambda$ in which the $J = 1$ hypernuclear state can arise from coupling the Λ to either the ground state of ^{7}Li ($J_N = \frac{3}{2}$) or the 478 keV first excited state ($J_N = \frac{1}{2}$). The amount that these possibi-

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[3] G. RACAH: *Phys. Rev.*, **62**, 438 (1942). Numerical values of these coefficients are now tabulated. See for example K. M. HOWELL: *Table of Wigner 6-j Symbols* (University of Southampton, Research Report U. S. 58-1, June 1958); M. ROTENBERG, R. BIVINS, N. METROPOLIS and B. A. WOOTEN: *The 3-j and 6-j Symbols* (Cambridge, Mass., 1959).

lities are admixed depends on $b = (E_t - E_s)$, which cannot be determined from the experimental data at the present stage of refinement. When the nuclear states are widely spaced in energy, equation (12) with $J_N = \bar{J}_N$ gives the binding energy directly. For j-j coupling no mixing was possible since there we had to calculate $\langle J_N \cdot j_\Lambda \rangle$ and J_N is diagonal in the nuclear spin.

In principle, when the data on binding energy become sufficiently good one should be able to determine both a and b and then one could not only make predictions about the ground state spins of the hyperfragments, but also could predict spins and excitation energies of excited levels. However, equations (8) and (12) are based on pure j-j and pure L-S coupling, respectively, so even if a and b could be deduced one should not expect to get more than qualitative agreement with experiment since the nuclear states involved are known to follow intermediate coupling.

2. – Branching ratios.

a) *Theory.* – The fact that parity is not conserved in the decay $\Lambda \to p + \pi^-$ is clearly borne out by the asymmetry [4] of the emitted π. In addition, the Berkeley group [5] has analyzed a large number of decay events and has concluded because of the large up-down asymmetry, that the spin of the Λ is $\frac{1}{2}$. Thus the interaction operative in the decay contains both a parity-conserving and parity-nonconserving part and may be written as [6]

$$H' = \left(s + \frac{p}{iq_\Lambda} \sigma \cdot \nabla \right), \tag{13}$$

where q_Λ is the momentum of the emitted π for free Λ decay, σ is the nucleon spin operator, and s and p are the decay amplitudes.

In order to calculate the transition probability for this decay when the Λ-particle is bound to a p-shell nucleus, we make the following assumptions:

1) The Λ-particle does not appreciably change the nucleon configuration.

2) The requisite nuclear wave functions are those determined for intermediate coupling by KURATH [7].

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[4] F. S. CRAWFORD jr., M. CRESTI, M. L. GOOD, K. GOTTSTEIN, E. M. LYMAN, F. T. SOLMITZ, M. L. STEVENSON and H. K. TICHO: *Phys. Rev.*, **108**, 1102 (1957); F. EISLER *et al.*: *Phys. Rev.*, **108**, 1353 (1957); L. LEIPUNER and R. ADAIR: *Phys. Rev.*, **109**, 1358 (1958).

[5] F. S. CRAWFORD, M. CRESTI, M. L. GOOD, M. L. STEVENSON and H. K. TICHO: *Phys. Rev. Lett.*, **2**, 114 (1959).

[6] See for example R. H. DALITZ: *Phys. Rev.*, **112**, 605 (1958).

[7] D. KURATH: *Phys. Rev.*, **101**, 216 (1956).

3) Recoil and center-of-mass effects are unimportant. (For the nuclei we are interested in, these lead to approximately 10% corrections to the matrix elements.)

With these approximations, the wave function of the initial state, in which the Λ-particle is bound to a nucleus, is given by

$$\psi_J^M = \sum_{m_1 M_1} C_{\frac{1}{2} J_i}(JM;\, mM_i)\, \varphi_{J_i T_i}^{M_i T_z} \mu_\Lambda \chi_{\frac{1}{2}}^m , \tag{14}$$

where μ_Λ is the radial wave function of the Λ-particle bound in an s-state to the nucleus; $\chi_{\frac{1}{2}}^m$ is the spin eigenfunction of the particle; $\varphi_{J_i T_i}^{M_i T_z}$ is the wave function of the initial nucleus, which has spin J_i and isotopic spin T_i (*i.e.*, in the case of the hypernucleus $^7\mathrm{Li}_\Lambda$, $\varphi_{J_i T_i}^{M_i T_z}$ would be the ground state eigenfunction of the ^{6}Li nucleus); and $C_{\frac{1}{2} J_i}(JM;\, mM_i)$ is the Clebsch-Gordan coefficient which insures that the spin $\frac{1}{2}$, and the nuclear spin, J_i, couple to total spin J.

The wavefunction of the final state is denoted by $\varphi_{J_f T_f}^{M_f M_T}$. (In the example quoted in the preceding paragraph this would be the wave function of one of the low-lying states in ^{7}Be.) To calculate the decay probability, it is convenient to introduce the concept of fractional-parentage coefficients and write the final-state wave function, which has N p-state nucloens in it, as

$$\psi_{J_f T_f}^{M_f M_T} = \sum_{\substack{j \bar{J} \bar{T} \\ \bar{m} \bar{M} \mu_1 \mu_2}} C_{j\bar{J}}(J_f M_f;\, \bar{m}\bar{M})\, C_{\frac{1}{2}\bar{T}}(T_f M_T;\, \mu_1 \mu_2) \langle \bar{J}\bar{T};\, j\tfrac{1}{2}\} J_f T_f \rangle \varphi_{\bar{J}\bar{T}}^{\bar{M}\mu_2} Y_j^{\bar{m}} \xi_{\frac{1}{2}}^{\mu_1} , \tag{15}$$

where the $\varphi_{\bar{J}\bar{T}}^{\bar{M}\mu_2}$ form a complete set of antisymmetric wave functions in which $(N-1)$ p-state nucleons couple their spin and isotopic spin to $\bar{J}$ and $\bar{T}$; Y_j^m is the space-spin wave function of the particle that has been split off; $\xi_{\frac{1}{2}}^{\mu_1}$ the isotopic spin function of the particle, and $\langle \bar{J}\bar{T};\, j\tfrac{1}{2}\} J_f T_f \rangle$ is the fractional-parentage coefficient. The sum over j ($j = \frac{3}{2}$ or $\frac{1}{2}$) is necessitated by the fact that the majority of p-shell nuclei follow neither j-j nor L-S coupling and hence either a $p_{\frac{3}{2}}$ or $p_{\frac{1}{2}}$ particle may be split off.

To keep track of the isotopic spin quantum numbers we arbitrarily endow the Λ-particle with the same isotopic spin as the proton. Thus equation (14) should be multiplied by $\xi_{\frac{1}{2}}^{-\frac{1}{2}}$ and equation (13) by $(1-\tau_3)/2$. The matrix element of H' in equation (13) can be simply calculated by the techniques introduced by Racah [3]. The expression is

$$\begin{aligned} \frac{1}{2J+1} \sum_{M M_f} \left| \langle J_f M_f | H' | JM \rangle \right|^2 &= N(2J_f+1)\, C_{\frac{1}{2} T_i}^2(T_f M_T;\, -\tfrac{1}{2}\, T_z) \langle \mu_p | j_1(kr) | \mu_\Lambda \rangle^2 \cdot \\ \cdot \Big\{ s^2 \Big| I_1 \sum_j (-1)^j \sqrt{2j+1} \langle J_i T_i;\, j\tfrac{1}{2}\} J_f T_f \rangle W(\tfrac{1}{2} j J J_f;\, 1 J_i) \Big|^2 &+ 18_p^2 \sum_\lambda C_{11}^2(\lambda 0;\, 00) \cdot \\ \cdot \Big| I_\lambda \sum_j \sqrt{2j+1} \langle J_i T_i;\, j\tfrac{1}{2}\} J_f T_f \rangle W(1\lambda\tfrac{1}{2}\tfrac{1}{2};\, 1j)\, W(\tfrac{1}{2} J j J_f;\, J_i \lambda) \Big|^2 \Big\} , \end{aligned} \tag{16}$$

where W is the Racah coefficient ([3]), k is the wave number of the emitted meson, N is the final number of p-state nucleons (for example, $N=3$ in the reaction $^7\mathrm{Li}_\Lambda \to {}^7\mathrm{Be}+\pi^-$) J_i and T_i, and J_f and T_f refer to initial and final nuclear states, respectively. The matrix element of the Bessel function $j_1(kr)$ is taken between the radial proton wavefunction and the Λ wavefunction and integrated over $r^2\,dr$. In the part of the matrix element that is proportional to p^2, the allowable values of λ are 0 and 2. These values correspond to the fact that either an s- or d-wave pion may be created by the gradient term in equation (1). The values of I_0, I_1, and I_2 depend on the interaction between the nucleus and the emitted π^-. If the interaction with the final state is neglected, the wave function of the π meson is $\exp[ik\cdot r]$ and $I_1=1$, $I_0=I_2=k/q_\Lambda$. Throughout most of this paper we shall neglect the interaction with the final state and return to it only in Section (*e*).

For most of the p-shell nuclei, a nuclear coupling scheme intermediate between j-j and L-S is most appropriate. In these cases one has to calculate the fractional-parentage coefficients directly from Kurath's wave function ([7]).

In the heavier nuclei j-j coupling is a good approximation. The above equations are still valid if one replaces the sum over j by a single term. The fractional-parentage coefficients have been tabulated by EDMONDS and FLOWERS ([8]).

On the other hand, at the beginning of the p-shell the low-lying states of the nuclei ^{6}Li, ^{7}Li, and ^{7}Be closely follows L-S coupling. For these nuclei it is more convenient to express the nuclear wave function in the L-S representation. The method of calculating the required matrix element is similar to the j-j case. The analogue to equation (16) for the L-S scheme is

$$(16a)\qquad \frac{1}{2J+1}\sum_{MM_f}\left|\langle J_f M_f|H'|JM\rangle\right|^2 = N(2L_f+1)(2S_f+1)(2J_i+1)(2J_f+1)\cdot$$

$$\cdot C^2_{\frac12 T_i}(T_f M_T;-\tfrac12 T_z)\langle\mu_\nu|j_1(kr)|\mu_\Lambda\rangle^2\langle L_i S_i T_i\}L_f S_f T_f\rangle^2\cdot$$

$$\cdot\Big\{s^2|I_1|^2 W^2(S_i S_f J_i J;\tfrac12 L_i)W^2(S_f J L_f 1;L_i J_f)+18p^2\sum_\lambda C^2_{11}(\lambda 0;00)\cdot$$

$$\cdot|I_\lambda\sum_\nu(2\nu+1)W(L_i\nu L_f J_f;S_f 1)W(J_i J\tfrac12 1;\tfrac12\nu)W(L_i J_i S_f\tfrac12;S_i\nu)\cdot W(1J1J_f;\nu\lambda)|^2\Big\},$$

where L_i, S_i, J_i and T_i refer to the total orbital angular momentum, total spin, their vector sum, and the total isotopic spin of the initial nuclear state, respectively. The subscript f indicates the analogous quantities in the final nuclear state. The other symbols are the same as those used in equation (16).

b) $^7\mathrm{Li}_\Lambda \to {}^7\mathrm{Be}+\pi^-$. – Since the ground state spin of ^{6}Li is unity, there are two possible values for the angular momentum of $^7\mathrm{Li}_\Lambda$: $\frac32$ or $\frac12$. The former

([8]) A. R. EDMONDS and B. H. FLOWERS: *Proc. Roy. Soc.* (*London*), A **214**, 515 (1952).

would correspond to the Λ and nuclear spins lining up; and if this is the state of lowest energy, it would show that the Λ-nucleon force was more attractive in the triplet spin state. On the other hand, if experiment shows that the ground-state spin of the hypernucleus is $J=\frac{1}{2}$, then obviously the force is stronger in the singlet spin state.

We shall concern ourselves with the decay of $^7\mathrm{Li}_\Lambda$ to the ground state and first excited state of ^{7}Be, which have spin $\frac{3}{2}$ and $\frac{1}{2}$, respectively. As stated earlier, these states are adequately represented in L-S coupling. The wave function of ^{6}Li is $^{31}S_1$ and for ^{7}Be it is $^{22}P_{\frac{3}{2}}$ in the ground state and $^{22}P_{\frac{1}{2}}$ in the first excited state. The transition probabilities (T.P.) for each initial spin state is calculated from equation (16a). The slight departure from unity of the ratio of densities of states in phase space is entirely negligible. In this case independent of whether the hypernucleus has spin $\frac{3}{2}$ or $\frac{1}{2}$ one finds

$$\frac{\text{T.P. to } \frac{1}{2} \text{ first excited state of } ^7\text{Be}}{\text{T.P. to } \frac{3}{2} \text{ ground state of } ^7\text{Be}} = \frac{1}{2}$$

so that no information on the spin dependence of the hypernuclear force can be inferred from a study of this decay.

c) $^{12}\mathrm{B}_\Lambda \to {}^{12}\mathrm{C}+\pi^-$. – The ground state spin of ^{11}B is $\frac{3}{2}$ which leads to the possibilities of $J=2$ or 1 for $^{12}\mathrm{B}_\Lambda$. The daughter nucleus, ^{12}C, has a ground state spin of zero and a first excited level of angular momentum two ([9]). In this case the nuclear wave functions follow intermediate coupling and the fractional-parentage coefficients needed in equation (16) have to be calculated from Kurath's wave functions ([7]). These are found to be

$$\langle \tfrac{3}{2}\,\tfrac{1}{2};\ \tfrac{3}{2}\,\tfrac{1}{2}\}\,0\ 0\rangle = 0.85\ ; \qquad \langle \tfrac{3}{2}\,\tfrac{1}{2};\ \tfrac{3}{2}\,\tfrac{1}{2}\}\,2\ 0\rangle = 0.002\ ;$$

and

$$\langle \tfrac{3}{2}\,\tfrac{1}{2};\ \tfrac{1}{2}\,\tfrac{1}{2}\}\,2\ 0\rangle = -0.36, \quad \text{for } L/K = 6.8 \text{ and } a/K = 4.5\ .$$

The phase space effects are included in what follows, and account for the factor (.91) which is shown explicitly.

1) If the ground state spin of $^{12}\mathrm{B}_\Lambda$ is 2, then one finds

$$\frac{\text{T.P. to } J=2 \text{ first exited state in } ^{12}\text{C}}{\text{T.P. to } J=0 \text{ ground state in } ^{12}\text{C}} = (.91)(2.5)\frac{0.129s^2 + 0.258p^2(k^2/q_\Lambda^2)}{0.723p^2(k'^2/q_\Lambda)^2}\ ,$$

which has a value that is always greater than 0.83, the minimum occurring when $s=0$. The ratio has the value 1.21 if it is assumed that $s^2 = p^2(k^2/q^2) = p^2(k'^2/q^2)$.

([9]) A compilation of properties of low-lying states in light nuclei is given by F. AJZENBERG-SELOVE and T. LAURITSEN: *Nucl. Phys.*, **11**, 1 (1959).

2) If the ground-state spin of $^{12}B_\Lambda$ is 1, the ratio is

$$\frac{\text{T.P. to } J=2 \text{ first excited state in } {}^{12}C}{\text{T.P. to } J=0 \text{ ground state in } {}^{12}C} = (.91)(.44) .$$

This ratio is independent of s^2 and p^2 since only the s part of equation (1) gives a contribution. This is accidental and depends on the fact that the fractional parentage coefficient $\langle \frac{3}{2}\,\frac{1}{2};\ \frac{3}{2}\,\frac{1}{2}\} 2\ 0\rangle \simeq 0$.

If an experiment determining these branching ratios shows that the ground state of $^{12}B_\Lambda$ has $J=2$, then one can conclude that there is a tendency for the Λ and nucleon spins to line up. This indicates that the triplet force is stronger. On the other hand, if $J=1$ is favored for the ground state, the singlet force is the more attractive one. Although at present there is only one event in which $^{12}B_\Lambda$ has been produced, when a sufficient number have been made the experiment should be reasonably simple since the first excited state of ^{12}C lies at 4.43 MeV.

d) *Other possibilities.* – In addition to $^7Li_\Lambda$ and $^{12}B_\Lambda$, which we have considered, there are other p-shell nuclei to which the hyperon may attach itself. Some of these are uninteresting from the point of view of our calculation since the ground state spins are almost certainly $\frac{1}{2}$. These include $^7He_\Lambda$ and $^9Be_\Lambda$ in which the Λ couples to a nucleus which has spin zero.

It could be argued that the ground state of the hypernucleus arises from coupling to an excited nuclear state. Assuming for simplicity that the nucleons obey j-j coupling, one can estimate from the binding energy formula, equation (8), what conditions this situation would impose on the Λ-nucleon interactions E_s and E_t. Using the fact that the first excited state in ^6He is 2^+ and has energy 1.71 MeV above ground, one concludes that if the $\frac{3}{2}$ state lies below the $\frac{1}{2}$ level we must have

$$E_t - E_s > 2\,(1.71)\ \text{MeV} .$$

The alternative, that the $\frac{5}{2}$ level lies below the $\frac{1}{2}$, implies that

$$E_s - E_t > 3\,(1.71)\ \text{MeV} .$$

Both these conclusions seem to be ruled out by the earlier calculations of DALITZ and DOWNS ([10]) who analyzed the binding energies of lighter hypernuclei. In the case of ^8Be the situation is even less likely since the first excited state lies at 2.90 MeV.

([10]) R. H. DALITZ and B. W. DOWNS: *Phys. Rev.*, **111**, 967 (1958).

One might ask whether a study of the decay ${}^{8}\mathrm{Li}_{\Lambda} \to {}^{8}\mathrm{Be}+\pi^{-}$ would be profitable. However, in this case [because of the low-lying (478 keV) $J=\frac{1}{2}$ state in ${}^{7}\mathrm{Li}$] there are three possible spins for the grouns state, $J=2$, 1, or 0. The state with $J=1$ can be obtained from coupling the Λ to either the ground state, $J=\frac{3}{2}$, or the excited state, $J=\frac{1}{2}$, the proper wave function being some linear combination of the two possibilities. Since the transition probabilities are quite sensitive to admixture, it is senseless to make a calculation until the correct mixing is known (*).

The decay of ${}^{9}\mathrm{Li}_{\Lambda}$, which probably has $J=\frac{5}{2}$ or $\frac{3}{2}$, to ${}^{9}\mathrm{Be}+\pi^{-}$ has not been studied because of the uncertainty in the spins of the excited states of the daughter nucleus. Only the ground state, $\frac{3}{2}^{-}$, is definitely known. The 2.43 MeV state has a small width and therefore high angular momentum, perhaps $\frac{5}{2}^{-}$ [which would agree with Kurath's calculation (7)]. Decay to the 1.75 MeV, $\frac{1}{2}^{+}$ state in ${}^{9}\mathrm{Be}$ is unlikely. This follows from the fact that the wave function for this level is probably predominantly made up of the ${}^{8}\mathrm{Be}$ ground state plus a particle in the $2s_{\frac{1}{2}}$ level. However, the matrix element, equation (4), arising from this would be identically zero since ${}^{8}\mathrm{Be}$ has $J=0$, $T=0$ while ${}^{8}\mathrm{Li}$ has $J=2$, $T=1$.

Finally the decay ${}^{11}\mathrm{B}_{\Lambda}$ (with spin $\frac{7}{2}$ or $\frac{5}{2}$) $\to {}^{11}\mathrm{C}+\pi^{+}$ has not been considered since even the ground state spin of ${}^{11}\mathrm{C}$ has not been determined with certainty.

Therefore, at the present moment the most promising p-shell hypernucleus, from the point of view of these calculations, seems to be ${}^{12}\mathrm{Be}_{\Lambda}$.

e) Effect of final-state interaction. – In the case in which the outgoing pion interacts with the ${}^{12}\mathrm{C}$ nucleus, we may no longer use the plane-wave approximation but should use the actual wave function of the meson moving in the interaction potential. The matrix element $\int \Psi^{*}_{J_f} (\exp[-ik\cdot r]) H' \Psi_J \mathrm{d}\tau$ should, therefore, be replaced by

$$\int \Psi^{*}_{J_f} \Phi^{*} H' \Psi_J \mathrm{d}\tau ,$$

where Φ satisfies the meson wave equation in which U, the interaction potential, is replaced by U^{*} and (11)

$$\Phi \xrightarrow[r\to\infty]{} \exp[ik\cdot r] + \frac{1}{r} f(\theta, \varphi) \exp[-ikr] .$$

(*) It should be noted that the nuclei ${}^{6}\mathrm{Li}$ and ${}^{11}\mathrm{B}$ both have first excited levels which lie at an energy > 2 MeV. Thus any admixture of excited states into the ground state wave functions of ${}^{7}\mathrm{Li}_{\Lambda}$ and ${}^{12}\mathrm{B}_{\Lambda}$ should be small. In fact, the 2.18 MeV level in ${}^{6}\mathrm{Li}$ has spin 3^{+} and coupling a Λ to this level can only lead to $J=\frac{7}{2}$ or $\frac{5}{2}$ and *not* $\frac{3}{2}$ or $\frac{1}{2}$.

(11) M. GELL-MANN and M. L. GOLDBERGER: *Phys. Rev.*, **91**, 398 (1953).

To estimate the effect of the final-state interaction on the branching ratios, we have assumed that U may be approximated by a complex square-well potential of radius a. In this case equations (16) and (16a) are still valid if we set

$$I_0 = (k/q_\Lambda)\cos\delta_0^* \exp[i\delta_0^*] + \sqrt{2/\pi}\,\frac{(\alpha+\beta)^{\frac{5}{2}}}{kq_\Lambda}\exp[i\delta_0^*]\exp[\tfrac{1}{2}k^2/(\alpha+\beta)]\cdot$$
$$\cdot\left\{\int_0^a r^2\,dr[3-(\alpha+\beta)r^2]\exp[-(\alpha+\beta)r^2/2]\,[c_0^* j_0(\varkappa^* r)-\cos\delta_0^* j_0(kr)]-\right.$$
$$\left.-\sin\delta_0^*\int_a^\infty r^2\,dr\,[3-(\alpha+\beta)r^2]\exp[-(\alpha+\beta)r^2/2]\,n_0(kr)\right\},$$

$$I_1 = \cos\delta_1^*\exp(i\delta_1^*) + \sqrt{2/\pi}\,\frac{(\alpha+\beta)^{\frac{5}{2}}}{k}\exp(i\delta_1^*)\exp[\tfrac{1}{2}k^2(\alpha+\beta)]\cdot$$
$$\cdot\left\{\int_0^a r^3\,dr\exp[-(\alpha+\beta)r^2/2][c_1^* j_1(\varkappa^* r)-\cos\delta_1^* j_1(kr)]-\right.$$
$$\left.-\sin\delta_1^*\int_a^\infty r^3\,dr\exp[-(\alpha+\beta)r^2/2]n_1(kr)\right\},$$

$$I_2 = (k/q_\Lambda)\cos\delta_2^*\exp(i\delta_2^*) + \sqrt{2/\pi}\,\frac{(\alpha+\beta)^{\frac{5}{2}}}{kq_\Lambda}\exp(i\delta_2^*)\exp[\tfrac{1}{2}k^2/(\alpha+\beta)]\cdot$$
$$\cdot\left\{\int_0^a r^4\,dr\exp[-(\alpha+\beta)r^2/2][c_2^* j_2(\varkappa^* r)-\cos\delta_2^* j_2(kr)]-\right.$$
$$\left.-\sin\delta_2^*\int_a^\infty r^4\,dr\exp[-(\alpha+\beta)r^2/2]n_2(kr)\right\},$$

where δ_l is the phase shift due to the interaction potential, j_l and n_l are the spherical Bessel and Neumann functions, $\varkappa$ is the wave number of the meson inside the complex potential, c_l is a constant which is determined by requiring continuity of the wave function at $r=a$, and the radial part of the Λ and p nucleon wave functions are proportional to $\exp[-\beta r^2/2]$ and $\exp[-\alpha r^2/2]$ respectively.

The results obtained by assuming a square-well potential of radius $2.5\cdot10^{-13}$ cm are given in Table I. For real U the I_l are increased since the interaction potential localizes the meson. The effect of the absorption tends to offset the effect of the localization, but even when this is included the overlap integrals still have values greater than those obtained in the absence of an

interaction with the final state. The value of U is similar to that used by BAKER, BYFIELD, and RAINWATER [12]. Although in their case a best fit to the data on π-meson scattering was obtained with a diffuse well. The value of β was estimated from the variational calculation given by DALITZ and DOWNS [10] and, on the assumption that the Λ-nucleon force is due to the exchange of 2 π-mesons, was found to be $\beta = 0.357 \cdot 10^{26}$ cm^{-2}. The value of α is that found from an analysis [13] of the electron scattering experiments, namely, $\alpha = 0.367 \cdot 10^{26}$ cm^{-2}.

TABLE I. – *Values of the phase shifts and* $|I_l|^2$ *for a square-well potential of radius* $2.5 \cdot 10^{-13}$ cm. *The wave-number of the outgoing meson is* $k = 0.597 \cdot 10^{13}$ cm^{-1}, $\alpha = 0.367 \cdot 10^{26}$ cm^{-2} *and* $\beta = 0.357 \cdot 10^{26}$ cm^{-2}.

U (MeV)	δ_0	δ_1	δ_2	$\lvert I_0\rvert^2$	$\lvert I_1\rvert^2$	$\lvert I_2\rvert^2$
0	0	0	0	$(k/q_\Lambda)^2$	1	$(k/q_\Lambda)^2$
-30	0.67	0.13	0	$3.13(k/q_\Lambda)^2$	1.59	$1.10(k/q_\Lambda)^2$
$-30-22i$	$0.72-0.51i$	$0.09-0.13i$	0	$1.76(k/q_\Lambda)^2$	1.18	$1.08(k/q_\Lambda)^2$

If the ground state spin of the $^{12}B_\Lambda$ hypernucleus is 2, the inclusion of the interaction with the final state changes the branching ratio for the decay so that

$$\frac{\text{T.P. to } J=2 \text{ first excited state in } {}^{12}C}{\text{T.P. to } J=0 \text{ ground state of } {}^{12}C} = (.91)\,\frac{0.064\,|I_1|^2 s^2 + 0.129\,|I_0|^2 p^2}{0.145\,|I_2|^2 p^2} =$$

$$= 2.96 \quad \text{if} \quad U = -30 \text{ MeV},$$

$$= 1.75 \quad \text{if} \quad U = (-30-22i) \text{ MeV},$$

which is to be compared with the value 1.21 found when $U = 0$. In these numerical estimates it is assumed that $s^2 = p^2(k^2/q_\Lambda^2) = p^2(k'^2/q_\Lambda^2)$.

On the other hand, if the ground state spin of $^{12}B_\Lambda$ is unity, the decay to both the excited and ground states of ^{12}C involves only $s^2|I_1|^2$ so that in the ratio, the influence of the final state interaction cancels out. This exact cancellation assumes that I_1 is the same whether the π is emitted in a transition to the ground state or first excited state of ^{12}C.

In calculating absolute decay rates for $^{12}B_\Lambda$ it may be convenient to write

[12] F. L. BAKER, H. BYFIELD and J. RAINWATER: *Phys. Rev.*, **112**, 1773 (1958).
[13] U. MEYER-BERKHOUT, K. W. FORD and A. E. S. GREEN: *Ann. Phys.*, **8**, 119 (1958).

the matrix element, equation (16), in the form,

$$(17) \qquad \frac{1}{2J+1}\sum_{MM_f}|\langle J_f M_f|H'|JM\rangle|^2 = \alpha_1(J,J_f)|I_1|^2 s^2 + [\alpha_0(J,J_f)|I_3|^2 + \alpha_2(J,J_f)|I_2|^2]p^2 .$$

The factors of α include all the geometrical effects, and are given in Table II. There are two values of $(k/q_\Lambda)^2$, the one to be used depending on whether the

TABLE II. – *Values of* α *appearing in eq.* (17). *The values are for the decay of* ^{12}B *only.*

$(J, J_f)=$	(2, 2)	(2, 0)	(1, 2)	(1, 0)
α_0	0.053	0	0	0
α_1	0.026	0	0.044	0.100
α_2	0	0.059	0	0

$^{12}B_\Lambda$ decay is to the ground state or to the first excited state of ^{12}C. They are, respectively, 1.45 and 1.26. The radial matrix element, which is also included in α, has the value 0.319.

* * *

We thank Professor R. H. DALITZ for pointing out the existence of the problems in p-shell hypernuclei, and for many interesting discussions.

RIASSUNTO (*)

Si studia la possibilità di determinare la dipendenza della forza del nucleone Λ con uno studio degli ipernuclei in cui la particella Λ è legata ad un nucleo avente un livello nucleare incompleto ($1p$). Si sottolinea che le attuali incertezze sui dati delle energie di legame degli ipernuclei permettono solo la determinazione della combinazione E_s+3E_t, in cui E_s ed E_t sono le energie di interazione del singoletto e del tripletto, rispettivamente. Si mostra che il decadimento degli ipernuclei con emissione π^- può fornire informazioni sulla dipendenza della forza dallo spin se viene misurato il rapporto di branching per il decadimento allo stato fondamentale ed al primo stato eccitato del nucleo derivato. In particolare, se lo spin del $^{12}B_\Lambda$ è 2, il decadimento al primo stato eccitato del ^{12}C è favorito per un fattore 1.2 rispetto al decadimento allo stato fondamentale. Però, se $J=1$ per questo ipernucleo, il decadimento allo stato fondamentale viene favorito per un fattore 2.4. Il cambiamento che l'interazione del mesone π col nucleo ^{12}C produce nel rapporto di branching viene stimato usando un pozzo di potenziale quadrato complesso. L'inclusione dell'interazione aumenta in modo apprezzabile il rapporto di branching nel caso di uno spin ipernucleare 2, ma non fa cambiare questo rapporto nel caso di $J=1$. D'altra parte, per $^7Li_\Lambda$ il rapporto di branching per il decadimento allo stato fondamentale ed al primo stato eccitato del ^{7}Be è indipendente dallo spin dell'ipernucleo.

(*) *Traduzione a cura della Redazione.*

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Correlation Theory of Stationary Electromagnetic Fields.

Part I – The Basic Field Equations (*).

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Summary. — In recent papers a class of second order correlation tensors was introduced and certain differential equations which govern their propagation were postulated. These correlation tensors, which may be regarded as natural generalizations of functions used in the analysis of partially coherent optical wavefields, characterize the correlations which exist between the electromagnetic field vectors at any two points in the field, at any two instants of time. In the present paper a derivation of the basic differential equations is presented; and it is shown that the two sets into which the equations naturally split are not independent, but in fact follow from each other as a consequence of certain symmetry properties which the correlation tensors exhibit.

1. – Introduction.

The researches carried out in recent years in the field of coherence theory of optics have indicated the possibility of formulating rigorous laws for the propagation of second order correlation functions of stationary electromagnetic fields. These functions (in general 3×3 Cartesian tensors) characterize the

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correlations between the electromagnetic field vectors at any two points in the field, at any two instants of time.

The correlation tensors were introduced in connection with an attempt to formulate a large branch of optics in terms of measurable quantities (1). The reason for attempting such a formulation is the fact, that in the great majority of optical experiments the quantities which are measured are not the field vectors themselves. These vectors vary so rapidly that no detector can record their detail behaviour (*). One can usually only measure the time averages of certain quadratic functions (*e.g.* the electric energy density) of the field components. Unlike the field vectors themselves, the correlation tensors are closely related to quantities that can be measured in the majority of optical experiments.

The correlation tensors were used, in specialized form, in the analysis of several problems. In reference (2) and (3) the elementary theory of partial polarization (for quasi-monochromatic plane waves) was reformulated and brought into a form strictly analogous to that employed in connection with the scalar theory of partially coherent optical fields. The formulation was found to be intimately connected with a well known representation of partially polarized radiation in terms of Stokes parameters. A generalization of the Stokes parameters to waves that are not necessarily plane, was recently carried out within the framework of correlation analysis by P. Roman (4). More recently it was shown (5) that the effect of the interaction of a partially polarized quasi-monochromatic plane wave with certain commonly used optical devices may be described as a transformation of the correlation tensor, the transformation involving an operator which is characteristic of the device.

In one of the earlier papers (Wolf (6)), differential equations which govern the propagation of the correlation tensors were postulated (in dyadic form). In the present paper they are derived from Maxwell's equations by the direct application of tensor analysis and it is shown that the two sets of equations into which they naturally split up are not independent. In fact, each set follows from the other by the use of certain symmetry properties which the correlation tensors exhibit. Some invariance properties of these equations are

(1) E. Wolf: *Nuovo Cimento*, **12**, 884 (1954).

(*) It may be recalled that whilst the optical periods are of the order of 10^{-14} s, the resolving time of the best photomultipliers which are at present available is of the order of 10^{-9} s.

(2) M. Born and E. Wolf: *Principles of Optics* (London and New York, 1959),

(3) E. Wolf: *Nuovo Cimento*, **13**, 1165 (1959).

(4) P. Roman: *Nuovo Cimento*, **13**, 974 (1959).

(5) G. B. Parrent and P. Roman: *Nuovo Cimento*, **15**, 370 (1960).

(6) E. Wolf: Contribution in *Proc. Symposium on Astronom. Optics* (Amsterdam, 1956), p. 177.

noted. Finally the wave equations for the correlation tensors are derived from the basic set of equations.

In Part II (ROMAN and WOLF [7]), a number of conservation laws involving the correlation tensors will be established.

2. – The correlation tensors.

We consider a stationary electromagnetic field in vacuum. Let $\boldsymbol{E}^{(r)}(\boldsymbol{x}, t)$ and $\boldsymbol{H}^{(r)}(\boldsymbol{x}, t)$ represent the electric and magnetic field vectors at a point specified by position vector $\boldsymbol{x}$, at time t. These vectors are real, but it is useful to employ a complex representation which may be introduced as follows:

Let $E_j^{(r)}$ $(j = 1, 2, 3)$ be the Cartesian components of $\boldsymbol{E}^{(r)}$ with respect to a fixed system of rectangular axes and let (*)

$$(2.1) \qquad \begin{cases} E_j^{(r)}(\boldsymbol{x}, t; T) = E_j^{(r)}(\boldsymbol{x}, t) & \text{when } |t| \leqslant T, \\ = 0 & \text{when } |t| > T. \end{cases}$$

We develop $E_j^{(r)}$ into a real Fourier integral with respect to t, *i.e.* we write

$$(2.2) \qquad E_j^{(r)}(\boldsymbol{x}, t; T) = \int_0^\infty e_j(\boldsymbol{x}, \omega; T) \cos\left[\varphi_j(\boldsymbol{x}, \omega; T) - \omega t\right] \mathrm{d}\omega .$$

With $E_j^{(r)}$ we now associate the complex vector E_j obtained by replacing in the integrand the cosine function by the exponential function:

$$(2.3) \qquad E_j(\boldsymbol{x}, t; T) = \int_0^\infty e_j(\boldsymbol{x}, \omega; T) \exp\left[i\left[\varphi_j(\boldsymbol{x}, \omega; T) - \omega t\right]\right] \mathrm{d}\omega .$$

Then

$$(2.4) \qquad E_j(\boldsymbol{x}, t; T) = E_j^{(r)}(\boldsymbol{x}, t; T) + i E_j^{(i)}(\boldsymbol{x}, t; T) ,$$

(7) P. ROMAN and E. WOLF: *Nuovo Cimento*, **17**, 477 (1960).

(*) The truncated functions $E_j^{(r)}(\boldsymbol{x}, t; T)$ are introduced for reasons well known in the theory of random processes: since the field is assumed to be stationary, $E_j^{(r)}(\boldsymbol{x}, t)$ is not square integrable and hence does not necessarily possess a Fourier representation. On the other hand the truncated functions $E_j^{(r)}(\boldsymbol{x}, t; T)$ are amenable to Fourier analysis and yield results of physical interest when one proceeds to the limit $T \to \infty$ in the appropriate formulae.

where

$$E_j^{(i)}(\boldsymbol{x}, t; T) = \int_0^\infty e_j(\boldsymbol{x}, \omega; T) \sin\left[\varphi_j(\boldsymbol{x}, \omega; T) - \omega t\right] \mathrm{d}\omega \,. \tag{2.5}$$

It is seen that $E_j^{(r)}$ and $E_j^{(i)}$ are a pair of « conjugate functions » and, as is well known (cf. reference [8]), they are related by Hilbert's reciprocity relations,

$$\begin{cases} E_j^{(i)}(\boldsymbol{x}, t; T) = \dfrac{P}{\pi} \displaystyle\int_{-\infty}^{\infty} \frac{E_j^{(r)}(\boldsymbol{x}, t'; T)}{t' - t} \mathrm{d}t' \,, \\ E_j^{(r)}(\boldsymbol{x}, t; T) = -\dfrac{P}{\pi} \displaystyle\int_{-\infty}^{\infty} \frac{E_j^{(i)}(\boldsymbol{x}, t'; T)}{t' - t} \mathrm{d}t' \,, \end{cases} \tag{2.6}$$

where P denotes the Cauchy principal value at $t' = t$.

In a strictly similar manner we also associate a complex vector with the (truncated) magnetic field:

$$H_j(\boldsymbol{x}, t; T) = H_j^{(r)}(\boldsymbol{x}, t; T) + i H_j^{(i)}(\boldsymbol{x}, t; T) \,, \qquad (j = 1, 2, 3), \tag{2.7}$$

where $H_j^{(i)}$ is the conjugate function associated with the real magnetic field $H_j^{(r)}$.

A complex function such as E_j or H_j, the real and imaginary parts of which are conjugate functions is known as an *analytic signal* (*).

The correlation tensors referred to in the introduction may now be defined by formulae of the form

$$\mathscr{E}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \operatorname*{Lim}_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} E_j(\boldsymbol{x}_1, t+\tau; T) E_k^*(\boldsymbol{x}_2, t; T) \, \mathrm{d}t \,, \tag{2.8}$$

where the asterisk denotes the complex conjugate. Denoting the time averaging operation on the right hand side by sharp brackets, we have in all the following four correlation tensors:

$$\begin{cases} \mathscr{E}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \langle E_j(\boldsymbol{x}_1, t+\tau) \, E_k^*(\boldsymbol{x}_2, t) \rangle \,, \\ \mathscr{H}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \langle H_j(\boldsymbol{x}_1, t+\tau) \, H_k^*(\boldsymbol{x}_2, t) \rangle \,, \\ \mathscr{G}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \langle E_j(\boldsymbol{x}_1, t+\tau) \, H_k^*(\boldsymbol{x}_2, t) \rangle \,, \\ \tilde{\mathscr{G}}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \langle H_j(\boldsymbol{x}_1, t+\tau) \, E_k^*(\boldsymbol{x}_2, t) \rangle \,. \end{cases} \tag{2.9}$$

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(8) E. C. Titchmarsh: *Introduction to the Theory of Fourier Integrals*, 2nd Ed. (Oxford, 1948), chap. 5.

(*) References to papers dealing with properties of analytic signals will be found in Born and Wolf (2), Section 10·2.

The physical significance of these tensors is clear: they represent the correlations which exist between the field vectors at two typical points ($\boldsymbol{x}_1$ and $\boldsymbol{x}_2$), at time instants separated by a time interval τ.

Although the correlation tensors are in general complex, they are very closely related to the real correlation tensors associated with the physical (real) fields. To see this we separate the real and imaginary parts in each of the four tensors (2.9) and denote them by superscripts (r) and (i) respectively, *i.e.* we write

$$\mathscr{E}_{jk} = \mathscr{E}^{(r)}_{jk} + i\mathscr{E}^{(i)}_{jk}, \tag{2.10}$$

etc., where $\mathscr{E}^{(r)}_{jk}$ and $\mathscr{E}^{(i)}_{jk}$ are real. It then follows from formulae (VII) and (VI) established in the Appendix that

$$\left\{\begin{aligned} \mathscr{E}^{(r)}_{jk} &= 2\langle E^{(r)}_j(\boldsymbol{x}_1, t+\tau)\, E^{(r)}_k(\boldsymbol{x}_2, t)\rangle \\ &= 2\langle E^{(i)}_j(\boldsymbol{x}_1, t+\tau)\, E^{(i)}_k(\boldsymbol{x}_2, t)\rangle, \\ \mathscr{E}^{(i)}_{jk} &= 2\langle E^{(i)}_j(\boldsymbol{x}_1, t+\tau)\, E^{(r)}_k(\boldsymbol{x}_2, t)\rangle \\ &= -2\langle E^{(r)}_j(\boldsymbol{x}_1, t+\tau)\, E^{(i)}_k(\boldsymbol{x}_2, t)\rangle, \end{aligned}\right. \tag{2.11}$$

with similar expressions for the real and imaginary parts of the other correlation tensors. In particular we see that the real part of each correlation tensor is twice the correlation tensor formed by the components of the real field vectors. Moreover it is not difficult to show that $\mathscr{E}$ itself is an analytic signal, so that $\mathscr{E}^{(r)}_{jk}$ and $\mathscr{E}^{(i)}_{jk}$ bear the same relationship to each other as does $E^{(r)}_j$ and $E^{(i)}_j$, *i.e.* they form a pair of conjugate functions:

$$\left\{\begin{aligned} \mathscr{E}^{(i)}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) &= \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{\mathscr{E}^{(r)}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau')}{\tau'-\tau}\,\mathrm{d}\tau', \\ \mathscr{E}^{(r)}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) &= -\frac{P}{\pi}\int_{-\infty}^{\infty} \frac{\mathscr{E}^{(i)}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau')}{\tau'-\tau}\,\mathrm{d}\tau'. \end{aligned}\right. \tag{2.12}$$

(Strictly similar result holds, of course, for the real and imaginary parts of $\mathscr{H}$, $\mathscr{G}$ and $\mathscr{F}$). The proof of (2.12) is identical with that given in connection with the optical scalar theory of partial coherence (cf. Born and Wolf [2], § 10·3.2).

We note that the time averaged electric and magnetic energy densities, denoted by W_{el} and W_{mag} respectively, can easily be expressed in terms of

the correlation tensors. We have

$$\langle W_{\mathrm{el}} \rangle = \frac{1}{8\pi} \langle \boldsymbol{E}^{(r)2} \rangle , \tag{2.13}$$

where $\boldsymbol{E}^{(r)} = \boldsymbol{E}^{(r)}(\boldsymbol{x}, t)$. Now according to (2.4),

$$\langle \boldsymbol{E} \cdot \boldsymbol{E}^* \rangle = \langle \boldsymbol{E}^{(r)2} \rangle + \langle \boldsymbol{E}^{(i)2} \rangle + 2i \langle \boldsymbol{E}^{(r)} \cdot \boldsymbol{E}^{(i)} \rangle . \tag{2.14}$$

The first two terms on the right are, according to (2.11), equal to each other, and the last term vanishes since all the other terms are real. Hence $\langle \boldsymbol{E} \cdot \boldsymbol{E}^* \rangle = 2\langle \boldsymbol{E}^{(r)2} \rangle$ and (2.13) becomes

$$\langle W_{\mathrm{el}} \rangle = \frac{1}{16\pi} \langle \boldsymbol{E} \cdot \boldsymbol{E}^* \rangle = \frac{1}{16\pi} \operatorname{Sp} \mathscr{E}(\boldsymbol{x}, \boldsymbol{x}, 0) , \tag{2.15}$$

where Sp denotes the spur (trace). Similarly

$$\langle W_{\mathrm{m}} \rangle = \frac{1}{16\pi} \operatorname{Sp} \mathscr{H}(\boldsymbol{x}, \boldsymbol{x}, 0) . \tag{2.16}$$

Next consider the time average of the energy flow (Poynting vector) $\boldsymbol{S}$:

$$\langle \boldsymbol{S} \rangle = \frac{c}{4\pi} \langle \boldsymbol{E}^{(r)} \wedge \boldsymbol{H}^{(r)} \rangle . \tag{2.17}$$

Now if $\mathscr{R}$ denotes the real part, we have

$$\begin{aligned} \mathscr{R}\langle \boldsymbol{E} \wedge \boldsymbol{H}^* \rangle &= \mathscr{R}\langle (\boldsymbol{E}^{(r)} + i\boldsymbol{E}^{(i)}) \wedge (\boldsymbol{H}^{(r)} - i\boldsymbol{H}^{(i)}) \rangle \\ &= 2\langle \boldsymbol{E}^{(r)} \wedge \boldsymbol{H}^{(r)} \rangle , \end{aligned}$$

if relations of the type $\langle E_j^{(r)} H_k^{(r)} \rangle = \langle E_j^{(i)} H_k^{(i)} \rangle$, which follow from formula (VI) of the Appendix, are used. Hence (2.17) may be written as

$$\langle \boldsymbol{S} \rangle = \frac{c}{8\pi} \mathscr{R}\langle \boldsymbol{E} \wedge \boldsymbol{H}^* \rangle , \tag{2.18}$$

or, taking the components,

$$\left| \begin{aligned} \langle S_j \rangle &= \frac{c}{8\pi} \mathscr{R}\{\langle E_k H_l^* - E_l H_k^* \rangle\} \\ &= \frac{c}{8\pi} \mathscr{R}\{\mathscr{G}_{kl}(\boldsymbol{x}, \boldsymbol{x}, 0) - \mathscr{G}_{lk}(\boldsymbol{x}, \boldsymbol{x}, 0)\} \\ &= \frac{c}{8\pi} \mathscr{R}\{\mathscr{G}_{kl}(\boldsymbol{x}, \boldsymbol{x}, 0) - \tilde{\mathscr{G}}_{kl}(\boldsymbol{x}, \boldsymbol{x}, 0)\} , \end{aligned} \right. \tag{2.19}$$

with $(j, k, l) = (1, 2, 3)$ or cycl.

3. – The basic differential equations.

Since the electric and magnetic vectors are related by Maxwell's equations, the correlation tensors are evidently not independent of each other. We shall now derive the relations which exist between them.

Let ε_{jkl} be the completely antisymmetric unit tensor of Levi-Civita, *i.e.* ε_{jkl} is $+1$ or -1 according as the subscripts (j, k, l) are an even or an odd permutation of $(1, 2, 3)$ and $\varepsilon_{jkl} = 0$ when two suffices are equal. The components of the vector product of any two vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ may then be expressed in the form

$$(\boldsymbol{V} \wedge \boldsymbol{W})_j = \varepsilon_{jkl} V_k W_l \,. \tag{3.1}$$

In particular, $\boldsymbol{V}$ may represent the operator $\nabla \equiv \partial/\partial x_k \equiv \partial_k$. Then

$$(\nabla \wedge \boldsymbol{W})_j = \varepsilon_{jkl} \partial_k W_l \,. \tag{3.2}$$

Since we shall be dealing with functions which depend on the position of two points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, we must distinguish between two differential operators acting on space co-ordinates. We shall use superscript 1 or 2 according whether the operator acts on the co-ordinates of $\boldsymbol{x}_1$, or $\boldsymbol{x}_2$, *i.e.*

$$\partial_k^1 \equiv \frac{\partial}{\partial x_1^k}, \qquad \partial_k^2 \equiv \frac{\partial}{\partial x_2^k} \qquad (k = 1, 2, 3). \tag{3.3}$$

Now Maxwell's equations for the electromagnetic field in free space may be expressed in the form (*)

$$\varepsilon_{jkl} \partial_k^1 E_l(\boldsymbol{x}_1, t_1) + \frac{1}{c} \frac{\partial}{\partial t_1} H_j(\boldsymbol{x}_1, t_1) = 0 \,, \tag{3.4}$$

$$\varepsilon_{jkl} \partial_k^1 H_l(\boldsymbol{x}_1, t_1) - \frac{1}{c} \frac{\partial}{\partial t_1} E_j(\boldsymbol{x}_1, t_1) = 0 \,, \tag{3.5}$$

(*) It is assumed here that not only the real fields $\boldsymbol{E}^{(r)}$, $\boldsymbol{H}^{(r)}$ but also the associated complex fields $\boldsymbol{E} = \boldsymbol{E}^{(r)} + i\boldsymbol{E}^{(i)}$, $\boldsymbol{H} = \boldsymbol{H}^{(r)} + i\boldsymbol{H}^{(i)}$ where $\boldsymbol{E}^{(i)}$ and $\boldsymbol{H}^{(i)}$ are functions conjugate to $\boldsymbol{E}^{(r)}$ and $\boldsymbol{H}^{(r)}$ respectively, satisfy Maxwell's equations. That this is so may be readily shown by taking the Hilbert transform of Maxwell's equations for the real fields $\boldsymbol{E}^{(r)}$, $\boldsymbol{H}^{(r)}$ and using the fact that if two functions are Hilbert transforms of each other so are their derivatives. One then finds that $\boldsymbol{E}^{(i)}$, $\boldsymbol{H}^{(i)}$ and hence the complex fields $\boldsymbol{E}$, $\boldsymbol{H}$ also obey Maxwell's equations.

and the divergence conditions may be written as

$$\partial_j^1 E_j(\boldsymbol{x}_1, t_1) = 0 , \tag{3.6}$$

$$\partial_j^1 H_j(\boldsymbol{x}_1, t_1) = 0 . \tag{3.7}$$

We multiply (3.4) by $E_m^*(\boldsymbol{x}_2, t_2)$ and obtain the relation

$$\varepsilon_{jkl}\,\partial_k^1 E_l(\boldsymbol{x}_1, t_1)\,E_m^*(\boldsymbol{x}_2, t_2) + \frac{1}{c}\frac{\partial}{\partial t_1} H_j(\boldsymbol{x}_1, t_1)\,E_m^*(\boldsymbol{x}_2, t_2) = 0 . \tag{3.8}$$

Let $t_1 = t_2 + \tau$ and keep t_2 fixed. Then $\partial/\partial t_1 = \partial/\partial\tau$ and (3.8) becomes, on taking the time average with respect to t_2, and finally writing t in place of t_2:

$$\varepsilon_{jkl}\,\partial_k^1 \langle E_l(\boldsymbol{x}_1, t+\tau)\,E_m^*(\boldsymbol{x}_2, t)\rangle + \frac{1}{c}\frac{\partial}{\partial\tau}\langle H_j(\boldsymbol{x}_1, t+\tau)\,E_m^*(\boldsymbol{x}_2, t)\rangle = 0 ,$$

or, in terms of the tensors $\mathscr{E}$ and $\tilde{\mathscr{G}}$ defined in (2.9),

$$\varepsilon_{jkl}\,\partial_k^1 \mathscr{E}_{lm} + \frac{1}{c}\frac{\partial}{\partial\tau}\tilde{\mathscr{G}}_{jm} = 0 . \tag{3.9a}$$

In a similar way, if we multiply (3.4) by $H_m^*(\boldsymbol{x}_2, t_2)$ and apply the same procedure, we obtain the equation

$$\varepsilon_{jkl}\,\partial_k^1 \mathscr{G}_{lm} + \frac{1}{c}\frac{\partial}{\partial\tau}\mathscr{H}_{jm} = 0 . \tag{3.10a}$$

From (3.5) we obtain in the same way

$$\varepsilon_{jkl}\,\partial_k^1 \tilde{\mathscr{G}}_{lm} - \frac{1}{c}\frac{\partial}{\partial\tau}\mathscr{E}_{jm} = 0 , \tag{3.11a}$$

and

$$\varepsilon_{ikl}\,\partial_k^1 \mathscr{H}_{lm} - \frac{1}{c}\frac{\partial}{\partial\tau}\mathscr{G}_{jm} = 0 . \tag{3.12a}$$

In a similar manner, the divergence conditions (3.6) and (3.7) yield the equations

$$\partial_j^1 \mathscr{E}_{jk} = 0 , \tag{3.13a}$$

$$\partial_j^1 \mathscr{G}_{jk} = 0 , \tag{3.14a}$$

$$\partial_j^1 \tilde{\mathscr{G}}_{jk} = 0 , \tag{3.15a}$$

$$\partial_j^1 \mathscr{H}_{jk} = 0 . \tag{3.16a}$$

There is another set of equations which involve the operator ∂_k^2. It may be conveniently obtained by starting from equations which are the complex conjugates of the « complex » Maxwell equations (3.4) and (3.5), after replacing $\boldsymbol{x}_1$, t_1 and ∂_k^1 by $\boldsymbol{x}_2$, t_2 and ∂_k^2 respectively. If these equations are then multiplied by $E_m(\boldsymbol{x}_1, t_1)$ or $H_m(\boldsymbol{x}_1, t_1)$ and time averaging is performed as before, one obtains:

$$\varepsilon_{jkl}\,\partial_k^2\,\mathscr{E}_{ml} - \frac{1}{c}\frac{\partial}{\partial\tau}\,\mathscr{G}_{mj} = 0\,, \tag{3.9b}$$

$$\varepsilon_{jkl}\,\partial_k^2\,\tilde{\mathscr{G}}_{ml} - \frac{1}{c}\frac{\partial}{\partial\tau}\,\mathscr{H}_{mj} = 0\,, \tag{3.10b}$$

$$\varepsilon_{jkl}\,\partial_k^2\,\mathscr{G}_{ml} + \frac{1}{c}\frac{\partial}{\partial\tau}\,\mathscr{E}_{mj} = 0\,, \tag{3.11b}$$

$$\varepsilon_{jkl}\,\partial_k^2\,\mathscr{H}_{ml} + \frac{1}{c}\frac{\partial}{\partial\tau}\,\mathscr{G}_{mj} = 0\,. \tag{3.12b}$$

From the divergence conditions (3.6) and (3.7) we find in a similar way that

$$\partial_j^2\,\mathscr{E}_{kj} = 0\,, \tag{3.13b}$$

$$\partial_j^2\,\tilde{\mathscr{G}}_{kj} = 0\,, \tag{3.14b}$$

$$\partial_j^2\,\mathscr{G}_{kj} = 0\,, \tag{3.15b}$$

$$\partial_j^2\,\mathscr{H}_{kj} = 0\,. \tag{3.16b}$$

The differential equations (3.9*a*)–(3.16*a*) and (3.9*b*)–(3.16*b*) were postulated (in dyadic notation) by WOLF [6]. Before we investigate some consequences of these equations, we note that each of the two sets of equations is invariant under the transformation

$$\mathscr{E} \rightleftarrows -\mathscr{H}\,, \qquad \mathscr{G} \rightleftarrows \tilde{\mathscr{G}}\,. \tag{3.17}$$

Moreover the set (*a*) is also invariant under the transformation

$$\mathscr{E} \rightleftarrows \mathscr{G}\,, \qquad \mathscr{H} \rightleftarrows \tilde{\mathscr{G}}\,, \tag{3.18}$$

and the set (*b*) under the transformation

$$\mathscr{E} \rightleftarrows \tilde{\mathscr{G}}\,, \qquad \mathscr{H} \rightleftarrows \mathscr{G}\,. \tag{3.19}$$

4. – Equivalence of the sets of eqs. (*a*) and (*b*).

It will now be shown that the sets of equations (*a*) and (*b*) are not independent but follow from each other as a consequence of certain « symmetry properties » which the correlation tensors exhibit.

We have

$$\left\{\begin{aligned} \mathscr{E}^{*}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) &= \langle E^{*}_{j}(\boldsymbol{x}_1, t+\tau) E_k(\boldsymbol{x}_2, t)\rangle \\ &= \langle E^{*}_{j}(\boldsymbol{x}_1, t) E_k(\boldsymbol{x}_2, t-\tau)\rangle \\ &= \mathscr{E}_{kj}(\boldsymbol{x}_2, \boldsymbol{x}_1, -\tau)\,. \end{aligned}\right. \tag{4.1}$$

On going from the first to the second line we have made use of the fact that the field is stationary. In a strictly similar manner

$$\mathscr{H}^{*}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \mathscr{H}_{kj}(\boldsymbol{x}_2, \boldsymbol{x}_1, -\tau)\,, \tag{4.2}$$

whilst the remaining two correlation tensors are found to be related by the formula (*)

$$\tilde{\mathscr{G}}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \mathscr{G}^{*}_{kj}(\boldsymbol{x}_2, \boldsymbol{x}_1, -\tau)\,. \tag{4.3}$$

We are again dealing with functions of two spacial arguments (and of the time difference τ) and must now distinguish between two « ∂ » operators, according to whether each operates on the *first* or the *second* spacial argument. We distinguish these by writing $\partial^{(1)}$ in the first case and $\partial^{(2)}$ in the second case. Thus if $F \equiv F(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau)$ then

$$\partial^{(1)}_k = \frac{\partial F}{\partial x^k_1} = \partial^1_k \qquad (k = 1, 2, 3);$$

but if $G = G(\boldsymbol{x}_2, \boldsymbol{x}_1, \tau)$ then

$$\partial^{(1)}_k = \frac{\partial G}{\partial x^k_2} = \partial^2_k \qquad (k = 1, 2, 3),$$

etc. It then follows from (4.1) that

$$\partial^{(1)}_l \mathscr{E}^{*}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \partial^{(2)}_l \mathscr{E}_{kj}(\boldsymbol{x}_2, \boldsymbol{x}_1, -\tau)\,. \tag{4.4}$$

We also have from (4.1)

$$\frac{\partial}{\partial \tau} \mathscr{E}^{*}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \frac{\partial}{\partial \tau} \mathscr{E}_{kj}(\boldsymbol{x}_2, \boldsymbol{x}_1, -\tau)\,. \tag{4.5}$$

(*) There is a misprint in the corresponding formula (eq. (18), p. 183) in ref. ([6]).

From (4.2) one obtains, of course, strictly similar relations involving the tensor $\mathscr{H}$, whilst the corresponding relations involving $\tilde{\mathscr{G}}$ and $\mathscr{G}$, deduced from (4.3), are

$$\partial_l^{(1)} \tilde{\mathscr{G}}^*_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \partial_l^{(2)} \mathscr{G}_{kj}(\boldsymbol{x}_2, \boldsymbol{x}_1, -\tau)\,, \tag{4.6}$$

and

$$\frac{\partial}{\partial \tau} \tilde{\mathscr{G}}^*_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \frac{\partial}{\partial \tau} \mathscr{G}_{kj}(\boldsymbol{x}_2, \boldsymbol{x}_1, -\tau)\,. \tag{4.7}$$

Let us now take the complex conjugate of (3.9a):

$$\varepsilon_{jkl} \partial_k^{(1)} \mathscr{E}^*_{lm}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) + \frac{1}{c} \frac{\partial}{\partial \tau} \tilde{\mathscr{G}}^*_{jm}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = 0\,. \tag{4.8}$$

Using the relation (4.4) and (4.7) this equation transforms into

$$\varepsilon_{jkl} \partial_k^{(2)} \mathscr{E}_{ml}(\boldsymbol{x}_2, \boldsymbol{x}_1, -\tau) + \frac{1}{c} \frac{\partial}{\partial \tau} \mathscr{G}_{mj}(\boldsymbol{x}_2, \boldsymbol{x}_1, -\tau) = 0\,. \tag{4.9}$$

If next we use the fact that

$$\frac{\partial}{\partial \tau} \mathscr{G}_{mj}(\boldsymbol{x}_2, \boldsymbol{x}_1, -\tau) = -\frac{\partial}{\partial(-\tau)} \mathscr{G}_{mj}(\boldsymbol{x}_2, \boldsymbol{x}_1, -\tau)$$

and set $\boldsymbol{x}_2 = \boldsymbol{u}_1$, $\boldsymbol{x}_1 = \boldsymbol{u}_2$, $-\tau = \varphi$, (4.9) becomes

$$\varepsilon_{jkl} \partial_k^{(2)} \mathscr{E}_{ml}(\boldsymbol{u}_1, \boldsymbol{u}_2, \varphi) - \frac{1}{c} \frac{\partial}{\partial \varphi} \mathscr{G}_{mj}(\boldsymbol{u}_1, \boldsymbol{u}_2, \varphi) = 0\,, \tag{4.10}$$

and this is just equation (3.9b) in another notation.

A similar procedure applied to equations (3.10a), (3.11a) and (3.12a) transforms them into equations (3.10b), (3.11b) and (3.12b) respectively.

Next consider the « divergence equations » (3.13a)–(3.16a). We have from (3.13a), with the help of (4.4)

$$\partial_j^{(2)} \mathscr{E}_{kj}(\boldsymbol{x}_2, \boldsymbol{x}_1, -\tau) = 0\,. \tag{4.11}$$

If we substitute $\boldsymbol{x}_2 = \boldsymbol{u}_1$, $\boldsymbol{x}_1 = \boldsymbol{u}_2$, $-\tau = \varphi$, this equation reduces to

$$\partial_j^{(2)} \mathscr{E}_{kj}(\boldsymbol{u}_1, \boldsymbol{u}_2, \varphi) = 0 \tag{4.12}$$

and this is equation (3.13b). In the same way one derives from the equations (3.14a), (3.15a) and (3.16a) the equations (3.14b), (3.15b) and (3.16b).

We have now shown that the equations (3.9*b*)–(3.16*b*) follow directly from the equations (3.9*a*)–(3.16*a*). Conversely it may be shown that the « *a* » equations follow from the « *b* » equations. We may thus restrict further discussions to equations of one of these two sets only. We select the « *a* » set and will from now on as a rule omit the superscript 1 on ∂_j^1. Thus ∂_j will denote differentiation with respect to the j-th component ($j = 1, 2, 3$) of the spatial argument $\boldsymbol{x}_1$. In case of ambiguity we will, however, retain the superscript.

In analogy with Maxwell's equations one may also show that the « divergence equations » (3.13*a*)–(3.16*a*) follow from the main equations in the sense that if they hold for a given value $\tau = \tau_0$, they will hold for all values of τ. To see this we apply the operators $\partial_j^1 \equiv \partial_j$ to (3.11*a*) and find that

$$\varepsilon_{jkl}\,\partial_j\,\partial_k\,\tilde{\mathscr{G}}_{lm} - \frac{1}{c}\,\frac{\partial}{\partial\tau}\,\partial_j\,\mathscr{E}_{jm} = 0\,.$$

Since ε_{jkl} is antisymmetric in the indices j, k, the first term, which involves the symmetric operator $\partial_j\partial_k$ vanishes and it follows that

$$\partial_j\,\mathscr{E}_{jm} = \text{constant}\,. \tag{4.13}$$

If this constant is zero when $\tau = \tau_0$, then the expression on the left is zero for all τ, in agreement with (3,13*a*). The same holds, of course, for the quantities $\partial_j\,\mathscr{G}_{jm}$, $\partial_j\,\tilde{\mathscr{G}}_{jm}$ and $\partial_j\mathscr{H}_{jm}$, in agreement with (3.14*a*)–(3.16*a*).

5. – Wave equations.

We shall now show that each correlation tensor satisfies the wave equation. On applying the operators $(1/c)\;\partial/\partial\tau$ to (3.11*a*), we have

$$\varepsilon_{jkl}\,\partial_k\,\frac{1}{c}\,\frac{\partial}{\partial\tau}\,\tilde{\mathscr{G}}_{lm} = \frac{1}{c^2}\,\frac{\partial^2}{\partial\tau^2}\,\mathscr{E}_{jm}\,. \tag{5.1}$$

Next we substitute on the left hand side from (3.9*a*) and obtain

$$-\,\varepsilon_{jkl}\varepsilon_{lab}\,\partial_k\,\partial_a\,\mathscr{E}_{bm} = \frac{1}{c^2}\,\frac{\partial^2}{\partial\tau^2}\,\mathscr{E}_{jm}\,. \tag{5.2}$$

Now there is the identity (see, for example JEFFREYS [9], p. 73)

$$\varepsilon_{jkl}\varepsilon_{lab} = \varepsilon_{jkl}\varepsilon_{abl} = \delta_{ja}\delta_{kb} - \delta_{jb}\delta_{ka}\,, \tag{5.3}$$

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[9] H. JEFFREYS and B. S. JEFFREYS: *Methods of Mathematical Physics*, 2nd Ed. (Cambridge, 1950).

where δ is the Kronecker symbol. Hence (5.3) becomes

$$-\partial_b\partial_j \mathscr{E}_{bm} + \nabla^2 \mathscr{E}_{jm} = \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\mathscr{E}_{jm}\,, \tag{5.4}$$

where $\nabla^2 \equiv \partial_k\partial_k$ $(k = 1, 2, 3)$ is the wave operator. Now according to (3.13*a*), the first term on the left of (5.4) vanishes, so that one finally obtains the equation

$$\nabla^2 \mathscr{E}_{jm} = \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\mathscr{E}_{jm}\,. \tag{5.5}$$

Hence the $\mathscr{E}$-tensor, and similarly also each of the tensors $\mathscr{H}$, $\mathscr{G}$ and $\tilde{\mathscr{G}}$, satisfies the homogeneous wave equation, it being understood here that the wave operator acts on the first spacial argument. However, if the « symmetry relations » (4.4) and (4.5) are used, one also obtains homogeneous wave equations in which the wave operator acts on the second spacial argument. (Alternatively they may be derived from the « discarded » set (3.9*b*)–(3.16*b*)).

The wave equations for the correlation tensors were previously given by WOLF [1], who derived them by using the fact that each field vector itself satisfies the wave equation. Here we have shown that the wave equations for the correlation tensors follow directly from the basic set (3.9*a*)–(3.16*a*).

In Part II of this investigation a number of conservation laws which relate to the correlation tensors will be established.

APPENDIX

Some theorems on analytic signals.

In this Appendix we shall establish some theorems needed in the main text, which relate to averages involving products of analytic signals.

Let $F^{(r)}(t)$ and $G^{(r)}(t)$ be two real stationary fields and consider the correlation function

$$\langle F(t+\tau)\,G(t)\rangle = \operatorname*{Lim}_{T\to\infty}\frac{1}{2T}\int_{-\infty}^{\infty} F(t+\tau;\,T)\,G(t;\,T)\,\mathrm{d}t\,, \tag{A.1}$$

where $F(t;\,T)$ and $G(t;\,T)$ are the analytic signals associated with the truncated functions $F^{(r)}(t;\,T)$ and $G^{(r)}(t;\,T)$ (cf. (2.1)). To evaluate (A.1) we de-

velop the functions on the right hand side of (A.1) into Fourier integrals (*):

$$\text{(A.2)} \qquad \begin{cases} F(t;\,T) = \int_0^\infty f(\omega;\,T)\,\exp\,[-i\omega t]\,\mathrm{d}\omega\,, \\ G(t;\,T) = \int_0^\infty g(\omega;\,T)\,\exp\,[-i\omega t]\,\mathrm{d}\omega\,. \end{cases}$$

From (A.1) and (A.2)

$$\text{(A.3)} \qquad \langle F(t+\tau)\,G(t)\rangle = -\lim_{T\to\infty}\frac{1}{2T}\int_{-\infty}^{\infty}\mathrm{d}t\int_0^\infty\!\!\int_0^\infty f(\omega;\,T)\,g(\omega';\,T)\,\exp\,[-i\omega\tau]\,\exp\,[-i(\omega+\omega')t]\,\mathrm{d}\omega\,\mathrm{d}\omega'\,.$$

Changing the order of the integrations and using the fact that

$$\int_{-\infty}^{\infty}\exp\,[-i(\omega+\omega')t]\,\mathrm{d}t = 2\pi\delta(\omega+\omega')\,,$$

where δ is the Dirac delta function, it immediately follows that

$$\text{(A.4)} \qquad \langle F(t+\tau)\,G(t)\rangle = 0\,.$$

Let

$$\text{(A.5)} \qquad F = F^{(r)} + iF^{(i)}\,, \qquad G = G^{(r)} + iG^{(i)}\,,$$

($F^{(r)}$, $F^{(i)}$, $G^{(r)}$, $G^{(i)}$ all real). It then follows on substitution from (A.5) into (A.4) and on equating real and imaginary parts, that

$$\text{(A.6)} \qquad \begin{cases} \langle F^{(r)}(t+\tau)\,G^{(r)}(t)\rangle = \langle F^{(i)}(t+\tau)\,G^{(i)}(t)\rangle\,, \\ \langle F^{(i)}(t+\tau)\,G^{(r)}(t)\rangle = -\langle F^{(r)}(t+\tau)\,G^{(i)}(t)\rangle\,. \end{cases}$$

(*) We assume here that F and G contain components corresponding to positive frequencies only, but strictly similar analysis and results hold when each contains spectral components corresponding only to negative frequencies. However, the results are not valid when F and G are analytic signal of « opposite kinds », *i.e.* when one contains a spectrum of positive frequencies, the other a spectrum of negative frequencies.

Finally we note a useful formula relating to the correlation function $\langle F(t+\tau)\,G^*(t)\rangle$. We have immediately, on using (A.5) and (A.6) that

$$\langle F(t+\tau)\,G^*(t)\rangle = 2\langle F^{(r)}(t+\tau)\,G^{(r)}(t) + 2i\langle F^{(i)}(t+\tau)\,G^{(r)}(t)\rangle\,. \tag{A.7}$$

RIASSUNTO (*)

In scritti recenti è stata introdotta una classe di tensori di correlazione di secondo ordine e sono state postulate alcune equazioni differenziali che regolano la loro propagazione. Questi tensori di correlazione, che possono essere considerati generalizzazioni naturali di funzioni usate nell'analisi dei campi d'onda ottici parzialmente coerenti, caratterizzano le correlazioni esistenti fra i vettori del campo elettromagnetico in qualsiasi coppia di punti del campo, e in qualsiasi due istanti. In questo scritto si presenta una deduzione delle equazioni differenziali fondamentali; e si mostra che i due gruppi nei quali le equazioni si suddividono naturalmente non sono indipendenti, ma in effetti derivano uno dall'altro come conseguenze di alcune proprietà di simmetria possedute dai tensori di correlazione.

(*) *Traduzione a cura della Redazione.*

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Correlation Theory of Stationary Electromagnetic Fields.

Part II. – Conservation Laws (*).

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(ricevuto il 22 Aprile 1960)

Summary. — Two new second order space-time correlation tensors $\mathscr{W}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau)$ and $\mathscr{S}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau)$ are introduced, which are simple linear combinations of the correlation tensors discussed in Part I of this investigation ($\boldsymbol{x}_1$, $\boldsymbol{x}_2$ are position vectors of two points and τ a time delay). These new tensors are intimately related to certain generalizations of the (time averaged) energy density and the energy flow vector. Differential equations which $\mathscr{W}_{jk}$ and $\mathscr{S}_{jk}$ satisfy in free space are derived, and from them four new conservation laws are deduced. In the limit $\boldsymbol{x}_1 \to \boldsymbol{x}_2$, $\tau \to 0$ two of these laws reduce to the usual laws (in time averaged form) for the conservation of the energy and the momentum in an electromagnetic field. The other two laws reduce only to trivial identities in this limit, so that they have no analogy in the framework of the usual theory.

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1. – Introduction.

In Part I of this investigation (Roman and Wolf [1]), to be referred to as I), differential equations were derived, which govern the propagation of a

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(1) P. Roman and E. Wolf: *Nuovo Cimento*, **17**, 461 (1960).

class of certain second-order correlation tensors associated with stationary electromagnetic fields in free space. The tensors describe the correlation between the electromagnetic field vectors at any two points $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ in space, at two instants of time separated by a time interval τ, and they contain information about the coherence as well as the polarization properties of the field. Since the correlation tensors represent averages taken over time intervals which are long compared with the mean periods of the vibrations of the field, they are of particular interest for optics, where, because of the great rapidity of the vibrations, only such averages and not the instantaneous values can be determined from experiment.

Second order correlation functions are, of course, not adequate to characterize completely all types of stationary electromagnetic fields. However, in many cases of practical importance, and certainly in most cases encountered in optics, the conditions for the validity of the central limit theorem of the theory of probability are effectively satisfied. One may then assume that the underlying probability distributions are Gaussian and in such cases second order correlations completely characterize the statistical behaviour of the field. It is, therefore, of some practical interest to investigate the consequences of the « second-order » theory, the mathematical structure of which was laid down in Part I.

In the present paper two new second-order correlation tensors $\mathscr{H}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau)$ and $\mathscr{S}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau)$ are introduced, which are simple linear combinations of the correlation tensors discussed in I. These new tensors are intimately related to certain generalizations of the (time averaged) energy density and the energy flow vector. With the help of the results of I, differential equations satisfied by $\mathscr{H}_{jk}$ and $\mathscr{S}_{jk}$ can be immediately written down and from them new types of conservation laws are derived. Some of these new laws are shown to be generalizations of the usual laws (in time averaged form) for the conservation of the energy and the momentum in an electromagnetic field. The other laws are found to have no analogy within the framework of the usual theory, which, unlike the present one deals only with averages evaluated for the same field point $(\boldsymbol{x}_1 = \boldsymbol{x}_2)$, at the same instant of time $(\tau = 0)$.

2. – The energy coherence tensor, the flow coherence tensor and the associated tensorial conservation laws.

Consider a stationary electromagnetic field and let $E_j(\boldsymbol{x}, t)$, $H_j(\boldsymbol{x}, t)$ $(j = 1, 2, 3)$ be the Cartesian components of the complex analytic signals associated with the (real) electric and magnetic fields, at a point specified by the position vector $\boldsymbol{x}$, at time t. The four correlation tensors employed in Part I are then

defined by the formulae

$$(2.1)\qquad \begin{cases} \mathscr{E}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \langle E_j(\boldsymbol{x}_1, t+\tau) E_k^*(\boldsymbol{x}_2, t)\rangle , \\ \mathscr{H}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \langle H_j(\boldsymbol{x}_1, t+\tau) H_k^*(\boldsymbol{x}_2, t)\rangle , \\ \mathscr{G}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \langle E_j(\boldsymbol{x}_1, t+\tau) H_k^*(\boldsymbol{x}_2, t)\rangle , \\ \tilde{\mathscr{G}}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \langle H_j(\boldsymbol{x}_1, t+\tau) E_k^*(\boldsymbol{x}_2, t)\rangle , \end{cases}$$

$(j, k = 1, 2, 3)$, where asterisk denotes the complex conjugate and sharp bracket denote the time average.

It was shown that in vacuo, the four correlation tensors are related by the following set of differential equations:

$$(2.2)\qquad \varepsilon_{jkl}\partial_k \mathscr{E}_{lm} + \frac{1}{c}\frac{\partial}{\partial\tau}\tilde{\mathscr{G}}_{jm} = 0 ,$$

$$(2.3)\qquad \varepsilon_{jkl}\partial_k \mathscr{G}_{lm} + \frac{1}{c}\frac{\partial}{\partial\tau}\mathscr{H}_{jm} = 0 ,$$

$$(2.4)\qquad \varepsilon_{jkl}\partial_k \tilde{\mathscr{G}}_{lm} - \frac{1}{c}\frac{\partial}{\partial\tau}\mathscr{E}_{jm} = 0 ,$$

$$(2.5)\qquad \varepsilon_{jkl}\partial_k \mathscr{H}_{lm} - \frac{1}{c}\frac{\partial}{\partial\tau}\mathscr{G}_{jm} = 0 ,$$

$(j, m = 1, 2, 3)$. The correlation tensors also satisfy the subsidiary conditions

$$(2.6)\qquad \partial_j \mathscr{E}_{jk} = 0 ,$$

$$(2.7)\qquad \partial_j \mathscr{G}_{jk} = 0 ,$$

$$(2.8)\qquad \partial_j \tilde{\mathscr{G}}_{jk} = 0 ,$$

$$(2.9)\qquad \partial_j \mathscr{H}_{jk} = 0 ,$$

$(k = 1, 2, 3)$. The symbol ε_{jkl} in equations (2.2)–(2.5) is the completely antisymmetric unit tensor of Levi-Civita ($\varepsilon_{jkl} = \pm 1$ according as j, k, l are even or odd permutations of $1, 2, 3$ and $\varepsilon_{jkl} = 0$ when two suffices are equal) and the symbol $\partial_k = \partial_k^1$ denotes the differential operator $\partial/\partial x_1^k$. There is a second set of differential equations which the correlation tensor satisfies. These equations involve the differential operator $\partial_k^2 = \partial/\partial x_2^k$ and may be deduced from

equations (2.2)–(2.9) by making the substitutions

$$\partial_k \equiv \partial_k^1 \to \partial_k^2, \qquad \mathscr{G}_{lm} \rightleftarrows -\tilde{\mathscr{G}}_{lm}; \tag{2.10}$$

this step utilizes the symmetry properties which the correlation tensors possess.

It was also shown in I that the time averaged energy density $\langle W \rangle$ and the time averaged energy flow vector $\langle \boldsymbol{S} \rangle$ are given by

$$\langle W \rangle = \frac{1}{16\pi} \mathrm{Sp}\{\mathscr{E}(\boldsymbol{x}, \boldsymbol{x}, 0) + \mathscr{H}(\boldsymbol{x}, \boldsymbol{x}, 0)\}, \tag{2.11}$$

$$\langle S_l \rangle = \frac{c}{8\pi} \mathscr{R}\{\mathscr{G}_{jk}(\boldsymbol{x}, \boldsymbol{x}, 0) - \tilde{\mathscr{G}}_{jk}(\boldsymbol{x}, \boldsymbol{x}, 0)\}, \tag{2.12}$$

where Sp denotes the spur (trace), $(j, k, l) = (1, 2, 3)$ or cycl., c is the vacuum velocity of light and $\mathscr{R}$ denotes the real part.

The (time averaged) energy density and energy flow vector are quadratic functions of the field components taken at the same point $(\boldsymbol{x})$ in space and at the same instant of time $(\tau = 0)$. To generalize these quantities within the framework of our correlation analysis, one must evidently replace the arguments $(\boldsymbol{x}, \boldsymbol{x}, 0)$ by $(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau)$ in (2.11) and (2.12). This leads to the introduction of two new tensors, *viz.*

$$\left\{ \begin{aligned} \mathscr{W}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) &= \mathscr{E}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) + \mathscr{H}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) \\ &= \langle E_j(\boldsymbol{x}_1, t+\tau) E_k^*(\boldsymbol{x}_2, t) \rangle + \langle H_j(\boldsymbol{x}_1, t+\tau) H_k^*(\boldsymbol{x}_2, t) \rangle, \end{aligned} \right. \tag{2.13}$$

$$\left\{ \begin{aligned} \mathscr{S}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) &= \mathscr{G}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) - \tilde{\mathscr{G}}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) \\ &= \langle E_j(\boldsymbol{x}_1, t+\tau) H_k^*(\boldsymbol{x}_2, t) - \langle H_j(\boldsymbol{x}_1, t+\tau) E_k^*(\boldsymbol{x}_2, t) \rangle. \end{aligned} \right. \tag{2.14}$$

The tensor $\mathscr{W}_{jk}$ will be called the *energy coherence tensor* and $\mathscr{S}_{jk}$ the *flow coherence tensor*. These tensors may be regarded as generalizations of the *mutual coherence function* $\Gamma(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \langle V(\boldsymbol{x}_1, t+\tau) V^*(\boldsymbol{x}_2, t) \rangle$ which plays a central role in the scalar theory of partial coherence (cf. BORN and WOLF [2]).

If as in I superscripts r and i denote real and imaginary parts, one has

---

[2] M. BORN and E. WOLF: *Principles of Optics* (London and New York, 1959), chap. X.

from (2.13) and from the relations (2.11) of I (*)

$$(2.15)\qquad \begin{cases} \mathscr{W}^{(r)}_{jk} = 2\left\{\langle E^{(r)}_j(\boldsymbol{x}_1,\, t+\tau)\, E^{(r)}_k(\boldsymbol{x}_2,\, t)\rangle + \langle H^{(r)}_j(\boldsymbol{x}_1,\, t+\tau)\, H^{(r)}_k(\boldsymbol{x}_2,\, t)\rangle\right\}, \\ \mathscr{W}^{(i)}_{jk} = 2\left\{\langle E^{(i)}_j(\boldsymbol{x}_1,\, t+\tau)\, E^{(r)}_k(\boldsymbol{x}_2,\, t)\rangle + \langle H^{(i)}_j(\boldsymbol{x}_1,\, t+\tau)\, H^{(r)}_k(\boldsymbol{x}_2,\, t)\rangle\right\}. \end{cases}$$

In a similar way it follows that

$$(2.16)\qquad \begin{cases} \mathscr{S}^{(r)}_{jk} = 2\left\{\langle E^{(r)}_j(\boldsymbol{x}_1,\, t+\tau)\, H^{(r)}_k(\boldsymbol{x}_2,\, t)\rangle - \langle H^{(r)}_j(\boldsymbol{x}_1,\, t+\tau)\, E^{(r)}_k(\boldsymbol{x}_2,\, t)\rangle\right\}, \\ \mathscr{S}^{(i)}_{jk} = 2\left\{\langle E^{(i)}_j(\boldsymbol{x}_1,\, t+\tau)\, H^{(r)}_k(\boldsymbol{x}_2,\, t)\rangle - \langle H^{(i)}_j(\boldsymbol{x}_1,\, t+\tau)\, E^{(r)}_k(\boldsymbol{x}_2,\, t)\rangle\right\}. \end{cases}$$

Moreover, because of the relations of the form given by (2.12) of I it follows that $\mathscr{W}^{(r)}$ and $\mathscr{W}^{(i)}$, and also $\mathscr{S}^{(r)}$ and $\mathscr{S}^{(i)}$ form conjugate pairs *i.e.* they are related by Hilbert's reciprocity formulae:

$$(2.17)\qquad \begin{cases} \mathscr{W}^{(i)}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \dfrac{P}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{\mathscr{W}^{(r)}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau')}{\tau' - \tau}\, \mathrm{d}\tau', \\ \mathscr{W}^{(r)}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = -\dfrac{P}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{\mathscr{W}^{(i)}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau')}{\tau' - \tau}\, \mathrm{d}\tau', \end{cases}$$

$$(2.18)\qquad \begin{cases} \mathscr{S}^{(i)}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \dfrac{P}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{\mathscr{S}^{(r)}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau')}{\tau' - \tau}\, \mathrm{d}\tau', \\ \mathscr{S}^{(r)}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = -\dfrac{P}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{\mathscr{S}^{(i)}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau')}{\tau' - \tau}\, \mathrm{d}\tau'. \end{cases}$$

It follows from (2.11) and (2.13) that the time averaged energy density is given by

$$(2.19)\qquad \langle W\rangle = \frac{1}{16\pi}\,\mathrm{Sp}\,\mathscr{W}(\boldsymbol{x}, \boldsymbol{x}, 0)\,,$$

and from (2.12) and (2.14) one obtains for the components of the time averaged energy flow vector the expressions

$$(2.20)\qquad \langle S_l\rangle = \frac{c}{8\pi}\,\mathscr{R}\mathscr{S}_{jk}(\boldsymbol{x}, \boldsymbol{x}, 0)\,.$$

(*) Here the *first* of each of the expression for $\mathscr{E}^{(r)}_{jk}$, $\mathscr{E}^{(i)}_{jk}$ etc. given by (2.11) of I were used. Alternative but completely equivalent formulae for $\mathscr{W}^{(r)}_{jk}$ etc. may be obtained by using the second expression of each of the eq. (2.11).

Because the electric and magnetic vectors are related by Maxwell's equations, the correlation tensors $\mathscr{W}$ and $\mathscr{S}$ are also coupled by differential equations. To find these equations one only has to add (2.2) and (2.5) and use the defining equations (2.13) and (2.14). One then obtains

$$\varepsilon_{jkl}\,\partial_k \mathscr{W}_{lm} - \frac{1}{c}\frac{\partial}{\partial \tau}\mathscr{S}_{jm} = 0\,, \tag{2.21}$$

($j, m = 1, 2, 3$). Further, on substracting (2.4) from (2.3) and using (2.13) and (2.14) one obtains the equation

$$\varepsilon_{jkl}\,\partial_k \mathscr{S}_{lm} + \frac{1}{c}\frac{\partial}{\partial \tau}\mathscr{W}_{jm} = 0\,. \tag{2.22}$$

Moreover, from (2.6) and (2.9) it follows that

$$\partial_j \mathscr{W}_{jk} = 0\,. \tag{2.23}$$

Finally from (2.7) and (2.8) one has

$$\partial_j \mathscr{S}_{jk} = 0\,. \tag{2.24}$$

The set of equations (2.21)–(2.24) is formally analogous to Maxwell's equations for vacuo and is invariant under the transformation

$$\mathscr{W} \to \mathscr{S}\,, \qquad \mathscr{S} \to -\mathscr{W}\,. \tag{2.25}$$

If the symmetry relations of I, Section **4** are used, one obtains from (2.21)–(2.24) a second set of differential equations, involving in place of the differential operator $\partial_k \equiv \partial/\partial x_1^k$ the differential operator $\partial/\partial x_2^k$ taken with respect to the co-ordinates of the spatial argument $\boldsymbol{x}_2$. Formally the second set may be obtained from equations (2.21)–(2.24) on making the substitution

$$\partial_k \equiv \partial_k^1 \to \partial_k^2\,, \qquad \mathscr{S} \to -\mathscr{S}\,. \tag{2.26}$$

These equations may, of course, also be obtained directly from the equations (3.9*b*)–(3.16*b*) of Part I.

We have now obtained from the original set of equations (2.2)–(2.9) which couple the correlation tensors $\mathscr{E}$, $\mathscr{H}$, $\mathscr{G}$ and $\mathscr{F}$ the set (2.21)–(2.24) which couples the correlation tensors $\mathscr{W}$ and $\mathscr{S}$. The equations (2.2)–(2.9) are, however, more general, since on transition to (2.21)–(2.24) part of the information which these equations contain was lost when taking the linear combinations.

On using the same procedure as was employed in the derivation of eq. (5.5) of Part I, one finds from (2.21)–(2.24) that $\mathscr{W}$ and $\mathscr{S}$ satisfy the homogeneous wave equation:

$$\nabla^2 \mathscr{W}_{jk} = \frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} \mathscr{W}_{jk} , \tag{2.27}$$

$$\nabla^2 \mathscr{S}_{jk} = \frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} \mathscr{S}_{jk} . \tag{2.28}$$

These equations may, of course, also be derived by taking the appropriata linear combinations of the previously derived equations for $\mathscr{E}$, $\mathscr{H}$, $\mathscr{G}$ and $\mathscr{T}$.

The equations (2.21) and (2.22), just like the original equations (2.2)–(2.5), have the form of *tensorial* conservation laws. For example, according to (2.21) $\mathscr{S}_{jm}$ is a conserved quantity, $\varepsilon_{jkl} \mathscr{W}_{lm}$ being the associated flow. Another class of conservation laws may readily be derived, some of which may be regarded as generalizations of the familiar conservation laws (in their time averaged form) for the energy and the momentum of the electromagnetic field. This class of conservation laws will be derived in the next sections.

3. – Scalar conservation laws.

On taking the trace of (2.21) one obtains

$$\begin{cases} \varepsilon_{jkl} \partial_k \mathscr{W}_{lj} - \dfrac{1}{c} \dfrac{\partial}{\partial \tau} \mathscr{S}_{jj} = 0 , \\ \partial^1_k (- \varepsilon_{klj} \mathscr{W}_{lj}) + \dfrac{1}{c} \dfrac{\partial}{\partial \tau} \operatorname{Sp} \mathscr{S} = 0 . \end{cases} \tag{3.1}$$

Here the superscript 1 has been attached to ∂_k to stress that the differential operator is to be taken with respect to the co-ordinates of the spatial argument $\boldsymbol{x}_1$.

In a strictly similar way one obtains from (2.22) (or more simply on transforming (3.1) in accordance to (2.25)),

$$\partial^1_k (\varepsilon_{klj} \mathscr{S}_{lj}) + \frac{1}{c} \frac{\partial}{\partial \tau} \operatorname{Sp} \mathscr{W} = 0 . \tag{3.2}$$

The formulae (3.1) and (3.2) are *scalar conservation laws*. According to (2.26) there are two further scalar conservation laws, which may be obtained by

replacing ∂_k^1 and ∂_k^2 and $\mathscr{S}$ and $-\mathscr{S}$ in (3.1) and (3.2). It will now be shown that (3.2) is a generalization of the usual energy law of the electromagnetic field (in its time averaged form). To this end consider the limiting form of (3.2) as the spatial argument $\boldsymbol{x}_2 \to \boldsymbol{x}_1$ ($= \boldsymbol{x}$ say) and also $\tau \to 0$. One has, first of all, from (2.14)

$$
\left.\begin{aligned}
\partial_k^1(\varepsilon_{klj}\mathscr{S}_{lj}) &= \partial_k^1\{\varepsilon_{klj}\langle \boldsymbol{E}_l(\boldsymbol{x}_1, t+\tau)\boldsymbol{H}_j^*(\boldsymbol{x}_2, t)\rangle - \varepsilon_{klj}\langle \boldsymbol{H}_l(\boldsymbol{x}_1, t+\tau)\boldsymbol{E}_j^*(\boldsymbol{x}_2, t)\rangle\} \\
&= \varepsilon_{klj}\langle \boldsymbol{H}_j^*(\boldsymbol{x}_2, t)\partial_k^1 \boldsymbol{E}_l(\boldsymbol{x}_1, t+\tau)\rangle - \varepsilon_{klj}\langle \boldsymbol{E}_j^*(\boldsymbol{x}_2, t)\partial_k^1 \boldsymbol{H}_l(\boldsymbol{x}_1, t+\tau)\rangle \\
&= \varepsilon_{klj}\langle \boldsymbol{H}_j^*(\boldsymbol{x}_2, t)\partial_k^1 \boldsymbol{E}_l(\boldsymbol{x}_1, t+\tau)\rangle + \varepsilon_{klj}\langle \boldsymbol{E}_l^*(\boldsymbol{x}_2, t)\partial_k^1 \boldsymbol{H}_j(\boldsymbol{x}_1, t+\tau)\rangle ,
\end{aligned}\right\} \tag{3.3}
$$

where on going from the second to the third line the dummy suffices l and j were interchanged and the relation $\varepsilon_{kjl} = -\varepsilon_{klj}$ was used. Hence

$$
\partial_k^1(\varepsilon_{klj}\mathscr{S}_{lj})\big|_{\boldsymbol{x},0} = \langle \varepsilon_{klj}[\boldsymbol{H}_j^*(\boldsymbol{x}, t)\partial_k \boldsymbol{E}_l(\boldsymbol{x}, t) + \boldsymbol{E}_l^*(\boldsymbol{x}, t)\partial_k \boldsymbol{H}_j(\boldsymbol{x}, t)]\rangle, \tag{3.4}
$$

where the symbol $|_{\boldsymbol{x},0}$ indicates that the quantity preceeding it is to be evaluated for $\boldsymbol{x}_1 = \boldsymbol{x}_2 = \boldsymbol{x}$, $\tau = 0$.

Next taking the real parts of (3.4), and using relations of the form (VII), given in Appendix of I, it follows that

$$
\left.\begin{aligned}
\mathscr{R}\,\partial_k^1(\varepsilon_{klj}\mathscr{S}_{lj})\big|_{\boldsymbol{x},0} &= 2\langle \varepsilon_{klj}(\boldsymbol{H}_j^{(r)}\partial_k \boldsymbol{E}_l^{(r)} + \boldsymbol{E}_l^{(r)}\partial_k \boldsymbol{H}_j^{(r)})\rangle \\
&= 2\langle (\varepsilon_{klj}\partial_k(\boldsymbol{E}_l^{(r)}\boldsymbol{H}_j^{(r)})\rangle \\
&= 2\langle \operatorname{div}(\boldsymbol{E}^{(r)} \wedge \boldsymbol{H}^{(r)})\rangle \\
&= -\frac{8\pi}{c}\langle \operatorname{div}\boldsymbol{S}(\boldsymbol{x}, t)\rangle ,
\end{aligned}\right\} \tag{3.5}
$$

where as before $\boldsymbol{S}$ denotes the energy flow vector.

The limiting form of the second term on the right hand side of (3.2) can be determined in a similar way. One has, from (2.13)

$$
\left.\begin{aligned}
\frac{\partial}{\partial\tau}\operatorname{Sp}\mathscr{W}\Big|_{\boldsymbol{x},0} &= \frac{\partial}{\partial\tau}\langle \boldsymbol{E}(\boldsymbol{x}_1, t+\tau)\cdot\boldsymbol{E}^*(\boldsymbol{x}_2, t) + \boldsymbol{H}(\boldsymbol{x}_1, t+\tau)\cdot\boldsymbol{H}^*(\boldsymbol{x}_2, t)\rangle\Big|_{\boldsymbol{x},0} \\
&= \left\langle \frac{\partial \boldsymbol{E}(\boldsymbol{x}, t)}{\partial t}\cdot\boldsymbol{E}^*(\boldsymbol{x}, t) + \frac{\partial \boldsymbol{H}(\boldsymbol{x}, t)}{\partial t}\cdot\boldsymbol{H}^*(\boldsymbol{x}, t)\right\rangle .
\end{aligned}\right\} \tag{3.6}
$$

Next we take the real part of (3.6). Using again the formula (VII) of Appendix of Part I, together with the fact that the time-derivative of an analytic signal

is again an analytic signal, one has from (3.6),

$$(3.7)\qquad \left\{\begin{aligned} \mathscr{R}\frac{\partial}{\partial\tau}\operatorname{Sp}\mathscr{W}\Big|_{x,0} &= 2\left\langle \frac{\partial \boldsymbol{E}^{(r)}}{\partial t}\cdot\boldsymbol{E}^{(r)} + \frac{\partial \boldsymbol{H}^{(r)}}{\partial t}\cdot\boldsymbol{H}^{(r)}\right\rangle \\ &= \left\langle \frac{\partial}{\partial t}\left(\boldsymbol{E}^{(r)2} + \boldsymbol{H}^{(r)2}\right)\right\rangle \\ &= 8\pi\left\langle \frac{\partial}{\partial t}\,W(\boldsymbol{x},t)\right\rangle , \end{aligned}\right.$$

where, as before W denotes the energy density of the field (*).

The meaning of the real part of the scalar conservation law (3.2), in the limiting case $\boldsymbol{x}_2 \to \boldsymbol{x}_1$ $(=\boldsymbol{x})$, $\tau \to 0$ is now seen on substitution from (3.5) and (3.7). One then obtains the formula

$$(3.8)\qquad \langle \operatorname{div}\boldsymbol{S}(\boldsymbol{x},t)\rangle + \left\langle \frac{\partial}{\partial t}\,W(\boldsymbol{x},t)\right\rangle = 0 ,$$

and this will be recognized as the time averaged form of the usual *energy conservation law* of the electromagnetic field in vacuum.

If one applies a similar limiting procedure to the other scalar conservation law, namely to (3.1), and uses Maxwell's equations, one obtains only a trivial identity. Thus (3.1) has no analogy in the usual theory which, in place of the second order correlation functions, deals with the products of field vectors considered at the same point in space and at the same instant of time.

Returning to the general case $(\boldsymbol{x}_2 \neq \boldsymbol{x}_1\ \tau \neq 0)$, the two conservation laws (3.1) and (3.2) may be written in a more explicit form. For this purpose we associate with each of the two tensors $\mathscr{W}$ and $\mathscr{S}$, a scalar and a vector defined by

$$(3.9a)\qquad \widehat{W}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau) = \langle \boldsymbol{E}(\boldsymbol{x}_1,t+\tau)\cdot\boldsymbol{E}^*(\boldsymbol{x}_2,t)\rangle + \langle \boldsymbol{H}(\boldsymbol{x}_1,t+\tau)\cdot\boldsymbol{H}^*(\boldsymbol{x}_2,t)\rangle ,$$

$$(3.9b)\qquad \widehat{\boldsymbol{W}}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau) = -c\{\langle \boldsymbol{E}(\boldsymbol{x}_1,t+\tau)\wedge\boldsymbol{E}^*(\boldsymbol{x}_2,t)\rangle + \langle \boldsymbol{H}(\boldsymbol{x}_1,t+\tau)\wedge\boldsymbol{H}^*(\boldsymbol{x}_2,t)\rangle\},$$

$$(3.9c)\qquad \widehat{\boldsymbol{S}}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau) = \langle \boldsymbol{E}(\boldsymbol{x}_1,t+\tau)\cdot\boldsymbol{H}^*(\boldsymbol{x}_2,t)\rangle - \langle \boldsymbol{H}(\boldsymbol{x}_1,t+\tau)\cdot\boldsymbol{E}^*(\boldsymbol{x}_2,t)\rangle ,$$

$$(3.9d)\qquad \widehat{\boldsymbol{S}}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau) = c\{\boldsymbol{E}(\boldsymbol{x}_1,t+\tau)\wedge\boldsymbol{H}^*(\boldsymbol{x}_2,t) - \langle \boldsymbol{H}(\boldsymbol{x}_1,t+\tau)\wedge\boldsymbol{E}^*(\boldsymbol{x}_2,t)\ \}.$$

(*) In a stationary field the average of a time-derivative, such as occurs in (3.7) or (4.6) is zero, but these terms are included here to show clearly the physical significance of the conservation laws.

The conservation laws (3.1) and (3.2) may then be expressed in the form

$$\operatorname{div}_1 \widehat{\boldsymbol{W}}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) + \frac{\partial}{\partial \tau} \widehat{S}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = 0 , \tag{3.10}$$

$$\operatorname{div}_1 \widehat{\boldsymbol{S}}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) + \frac{\partial}{\partial \tau} \widehat{W}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = 0 , \tag{3.11}$$

where div_1 is the divergence taken with respect to the co-ordinates of $\boldsymbol{x}_1$. There are, of course, two strictly similar conservation laws which may be formally derived from (3.10) and (3.11) on taking the divergence with respect to the co-ordinate of $\boldsymbol{x}_2$ and replacing $\hat{S}$ by $-\hat{S}$ and $\widehat{\boldsymbol{S}}$ by $-\widehat{\boldsymbol{S}}$, in accordance with (2.26).

The quantities $\widehat{W}$ and $\widehat{\boldsymbol{S}}$ are, apart from trivial factors, generalizations of the time averaged energy density $\langle W \rangle$ and the time averaged energy flow vector $\langle \boldsymbol{S} \rangle$ respectively. In fact

$$\langle W(\boldsymbol{x}, t) \rangle = \frac{1}{16\pi} \mathscr{R} \widehat{W}(\boldsymbol{x}, \boldsymbol{x}, 0) , \qquad \langle \boldsymbol{S}(\boldsymbol{x}, t) \rangle = \frac{1}{16\pi} \mathscr{R} \widehat{\boldsymbol{S}}(\boldsymbol{x}, \boldsymbol{x}, 0) . \tag{3.12}$$

The quantities $\widehat{\boldsymbol{W}}$ and $\hat{S}$, on the other hand, have no counterpart in the limiting case $\boldsymbol{x}_1 = \boldsymbol{x}_2$, $\tau = 0$, since evidently

$$\mathscr{R} \widehat{\boldsymbol{W}}(\boldsymbol{x}, \boldsymbol{x}, 0) = 0 , \qquad \mathscr{R} \hat{S}(\boldsymbol{x}, \boldsymbol{x}, 0) = 0 . \tag{3.13}$$

As we have already seen the conservation law (3.10), which involves $\widehat{\boldsymbol{W}}$ and $\hat{S}$ has no counterpart in this limiting case either.

4. – Vector conservation laws.

It is possible to derive from (2.21) and (2.22) two other conservation laws. One only has to multiply each of the two equations by ε_{snj} and use the identity (5.3) of I, namely

$$\varepsilon_{smj} \varepsilon_{jkl} = \delta_{sk} \delta_{ml} - \delta_{sl} \delta_{mk} . \tag{4.1}$$

One then obtains the following two *vector conservation laws*:

$$\partial_k^1 (\delta_{sk} \operatorname{Sp} \mathscr{W} - \mathscr{W}_{sk}) + \frac{1}{c} \frac{\partial}{\partial \tau} \varepsilon_{sjm} \mathscr{S}_{jm} = 0 , \qquad (s = 1, 2, 3), \tag{4.2}$$

$$\partial_k^1 [-(\delta_{sk} \operatorname{Sp} \mathscr{S} - \mathscr{S}_{sk})] + \frac{1}{c} \frac{\partial}{\partial \tau} \varepsilon_{sjm} \mathscr{W}_{jm} = 0 , \qquad (s = 1, 2, 3). \tag{4.3}$$

It will now be shown that (4.2) is a generalization of the conservation law (in its time averaged form) for the momentum of the electromagnetic field. From the defining equation (2.14) it follows that

$$\partial_k^1(\delta_{sk}\,\mathrm{Sp}\,\mathscr{W}-\mathscr{W}_{sk})-\langle\delta_{sk}\,\partial_k^1 E_j(\boldsymbol{x}_1,\,t+\tau)\,E_j^*(\boldsymbol{x}_2,\,t)\;-\;\partial_k^1 E_s(\boldsymbol{x}_1,\,t+\tau)\,E_k^*(\boldsymbol{x}_2,\,t)\rangle+\ldots$$

where the dots indicate two similar terms containing H in place of E. Hence

$$\partial_k(\delta_{sk}\,\mathrm{Sp}\,\mathscr{W}-\mathscr{W}_{sk})|_{\boldsymbol{x},0}=\langle\delta_{sk}E_j^*(\boldsymbol{x},\,t)\partial_k E_j(\boldsymbol{x},\,t)\rangle-\langle E_k^*(\boldsymbol{x},\,t)\partial_k E_s(\boldsymbol{x},\,t)\rangle+\ldots\,.$$

Let us subtract from the right hand side of this equation the expression $\langle E_s(\boldsymbol{x},\,t)\partial_k E_k^*(\boldsymbol{x},\,t)\rangle+\langle H_s(\boldsymbol{x},\,t)\partial_k H_s^*(\boldsymbol{x},\,t)\rangle$ which is identically zero because of the Maxwell equations $\mathrm{div}\,\boldsymbol{E}=\mathrm{div}\,\boldsymbol{H}=0$. One then obtains

$$\partial_k(\delta_{sk}\,\mathrm{Sp}\,\mathscr{W}-\mathscr{W}_{sk})|_{\boldsymbol{x},0}=\langle\delta_{sk}E_j^*\partial_k E_j)\rangle-\langle\partial_k(E_k^*E_s)\rangle+\ldots\,.$$

On taking the real part and using identities of the form (VII) of the Appendix in I, one finds that

$$\left\{\begin{aligned}\mathscr{R}\,\partial_s(\delta_{sk}\,\mathrm{Sp}\,\mathscr{W}-\mathscr{W}_{sk})|_{\boldsymbol{x},0}&=2\langle\delta_{sk}E_j^{(r)}\partial_k E_j^{(r)}-\partial_k(E_k^{(r)}E_s^{(r)})\rangle\\&=\langle\partial_k[\delta_{sk}(\boldsymbol{E}^{(r)^2}+\boldsymbol{H}^{(r)^2})-2(E_k^{(r)}E_s^{(r)}+H_k^{(r)}H_s^{(r)})]\rangle\\&=-8\pi\langle\partial_k T_{ks}(\boldsymbol{x},\,t)\rangle\,,\end{aligned}\right.\tag{4.4}$$

where $T_{ks}=T_{ks}(\boldsymbol{x},\,t)$, $(k,\,s=1,\,2,\,3)$, is the electromagnetic stress tensor:

$$T_{ks}=\frac{1}{4\pi}\left[(E_k^{(r)}E_s^{(r)}-\tfrac{1}{2}\,\delta_{ks}\boldsymbol{E}^{(r)^2})+(H_k^{(r)}H_s^{(r)}-\tfrac{1}{2}\,\delta_{ks}\boldsymbol{H}^{(r)^2})\right].\tag{4.5}$$

Next consider the second term on the right of (1.2). Using the defining equation (2.14), one has

$$\frac{\partial}{\partial\tau}\,\varepsilon_{sjm}\,\mathscr{S}_{jm}=\frac{\partial}{\partial\tau}\left\{\varepsilon_{sjm}\langle E_j(\boldsymbol{x}_1,\,t+\tau)\,H_m^*(\boldsymbol{x}_2,\,t)\rangle-\varepsilon_{sjm}\langle H_j(\boldsymbol{x}_1,\,t+\tau)\,E_m^*(\boldsymbol{x}_2,\,t)\rangle\right\}=$$

$$=\frac{\partial}{\partial\tau}\left\{\varepsilon_{sjm}\langle E_j(\boldsymbol{x}_1,\,t+\tau)\,H_m^*(\boldsymbol{x}_2,\,t)\rangle+\varepsilon_{sjm}\langle H_m(\boldsymbol{x}_1,\,t+\tau)\,E_j^*(\boldsymbol{x}_2,\,t)\rangle\right\},$$

where in the last term the dummy suffices j and m were interchanged and the relation $\varepsilon_{smj}=-\varepsilon_{sjm}$ was used. Hence

$$\frac{\partial}{\partial\tau}\,\varepsilon_{sjm}\,\mathscr{S}_{jm}\Big|_{\boldsymbol{x},0}=\varepsilon_{sjm}\left\langle\frac{\partial E_j(\boldsymbol{x},\,t)}{\partial t}\,H_m^*(\boldsymbol{x},\,t)+\frac{\partial H_m(\boldsymbol{x},\,t)}{\partial t}\,E_j^*(\boldsymbol{x},\,t)\right\rangle.$$

On taking the real parts and again using the formula (VII) of Appendix to Part I, (together with the fact that the time derivative of an analytic signal is again an analytic signal), one obtains the expression

$$
\left.
\begin{aligned}
\mathscr{R}\frac{\partial}{\partial\tau}\,\varepsilon_{sjm}\,\mathscr{S}_{jm}\Big|_{\boldsymbol{x},0} &= 2\varepsilon_{sjm}\left\langle \frac{\partial E_j^{(r)}}{\partial t}\,H_m^{(r)} + \frac{\partial H_m^{(r)}}{\partial t}\,E_j^{(r)}\right\rangle \\
&= 2\varepsilon_{sjm}\left\langle \frac{\partial}{\partial t}\,(E_j^{(r)}H_m^{(r)})\right\rangle \\
&= 8\pi c\left\langle \frac{\partial}{\partial t}\,P_s(\boldsymbol{x},t)\right\rangle ,
\end{aligned}
\right\} \tag{4.6}
$$

where $P_s = P_s(\boldsymbol{x}, t)$, $(s = 1, 2, 3)$, are the components of the momentum density:

$$
\boldsymbol{P} = \frac{1}{4\pi c}\,(\boldsymbol{E}^{(r)}\wedge\boldsymbol{H}^{(r)}). \tag{4.7}
$$

With the help of (4.4) and (4.6) it follows that the real part of the conservation law (4.2) reduces in the limit $\boldsymbol{x}_2\to\boldsymbol{x}_1$ $(=\boldsymbol{x})$, $\tau\to 0$, to

$$
\langle -\partial_k T_{ks}(\boldsymbol{x},t)\rangle + \left\langle \frac{\partial}{\partial t}\,P_s(\boldsymbol{x},t)\right\rangle = 0\,, \qquad (s=1,2,3), \tag{4.8}
$$

which is the usual *momentum conservation law* of the electromagnetic field in vacuum, in its time-averaged form.

A similar procedure applied to the other vectorial conservation law, namely (4.3), gives only a trivial identity in the limit $\boldsymbol{x}_2\to\boldsymbol{x}_1$, $\tau\to 0$. Thus (4.3) has no analogy in the usual theory.

Returning to the general case again $(\boldsymbol{x}_2\neq\boldsymbol{x}_1,\ \tau\neq 0)$, it is useful to rewrite (4.2) and (4.3) in a more explicit form. For this purpose it is convenient to introduce two new tensors $\hat{T}$ and $\hat{Q}$ associated with $\mathscr{W}$ and $\mathscr{S}$ respectively, defined by

$$
\begin{aligned}
\hat{T}_{ks}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau) &= \mathscr{W}_{ks}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau) + \mathscr{W}_{sk}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau) - \delta_{ks}\,\mathrm{Sp}\,\mathscr{W}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau) = \\
&= \langle E_k(\boldsymbol{x}_1,t+\tau)E_s^*(\boldsymbol{x}_2,t)\rangle + \langle E_s(\boldsymbol{x}_1,t+\tau)E_k^*(\boldsymbol{x}_2,t)\rangle - \delta_{ks}\langle \boldsymbol{E}(\boldsymbol{x}_1,t+\tau)\cdot\boldsymbol{E}^*(\boldsymbol{x}_2,t)\rangle + \\
&+ \langle H_k(\boldsymbol{x}_1,t+\tau)H_s^*(\boldsymbol{x}_2,t)\rangle + \langle H_s(\boldsymbol{x}_1,t+\tau)H_k^*(\boldsymbol{x}_2,t)\rangle - \delta_{ks}\langle \boldsymbol{H}(\boldsymbol{x}_1,t+\tau)\cdot\boldsymbol{H}^*(\boldsymbol{x}_2,t)\rangle\,,
\end{aligned} \tag{4.9}
$$

$$
\begin{aligned}
\hat{Q}_{ks}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau) &= c^2\{\mathscr{S}_{ks}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau) + \mathscr{S}_{sk}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau) - \delta_{ks}\,\mathrm{Sp}\,\mathscr{S}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau)\} = \\
&= c^2\{\langle E_k(\boldsymbol{x}_1,t+\tau)H_s^*(\boldsymbol{x}_2,t)\rangle + \langle E_s(\boldsymbol{x}_1,t+\tau)H_k^*(\boldsymbol{x}_2,t)\rangle - \delta_{ks}\langle \boldsymbol{E}(\boldsymbol{x}_1,t+\tau)\cdot\boldsymbol{H}^*(\boldsymbol{x}_2,t)\rangle \\
&- \langle H_k(\boldsymbol{x}_1,t+\tau)E_s^*(\boldsymbol{x}_2,t)\rangle - \langle H_s(\boldsymbol{x}_1,t+\tau)E_k^*(\boldsymbol{x}_2,t)\rangle + \delta_{ks}\langle \boldsymbol{H}(\boldsymbol{x}_1,t+\tau)\cdot\boldsymbol{E}^*(\boldsymbol{x}_2,t)\rangle\}\,.
\end{aligned} \tag{4.10}
$$

Further les us also introduce a vector $\hat{\boldsymbol{P}}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau)$ defined by

$$\widehat{\boldsymbol{P}}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \frac{1}{c^2} \widehat{\boldsymbol{S}}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) . \tag{4.11}$$

If use is also made of (2.23) and (2.24), the conservation laws (4.2) and (4.3) may then be written in the following form (*):

$$\partial_k^1 [- \hat{T}_{ks}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau)] + \frac{\partial}{\partial \tau} \hat{P}_s(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = 0 , \qquad (s = 1, 2, 3), \tag{4.12}$$

$$\partial_k^1 [- \hat{Q}_{ks}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau)] + \frac{\partial}{\partial \tau} \hat{W}_s(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = 0 , \qquad (s = 1, 2, 3). \tag{4.13}$$

Again there are two strictly similar conservation laws which may formally be derived from (4.11) and (4.12) by replacing ∂_k^1 by ∂_k^2 and by changing $\hat{\boldsymbol{S}}$ into $-\hat{\boldsymbol{S}}$ and $\hat{Q}$ into $-\hat{Q}$, in accordance with (2.26).

The tensor $\hat{T}_{ks}$ is, apart from a trivial factor, a generalization of the time averaged electromagnetic stress tensor $\langle T \rangle$. In fact

$$\langle T_{ks}(\boldsymbol{x}, t) \rangle = \frac{1}{16\pi} \mathscr{R} \hat{T}_{ks}(\boldsymbol{x}, \boldsymbol{x}, 0) ; \tag{4.14}$$

and apart from a trivial factor $\hat{\boldsymbol{P}}$ may be regarded as a generalization of the time averaged momentum density $\langle \boldsymbol{P}(\boldsymbol{x}, t) \rangle$:

$$\langle \boldsymbol{P}(\boldsymbol{x}, t) \rangle = \frac{1}{16\pi} \mathscr{R} \hat{\boldsymbol{P}}(\boldsymbol{x}, \boldsymbol{x}, 0) . \tag{4.15}$$

The tensors $\hat{Q}_{ks}$ (and as already noted also $\hat{\boldsymbol{W}}$) have no counterpart in the limiting case $\boldsymbol{x}_1 = \boldsymbol{x}_2$, $\tau = 0$, since

$$\mathscr{R} \hat{Q}_{ks}(\boldsymbol{x}, \boldsymbol{x}, 0) = 0 . \tag{4.16}$$

As has already been seen, the conservation law (4.13) has no analogue in this limiting case either.

(*) In eqs. (4.12) and (4.13) the symbols $\hat{P}_s$ and $\hat{W}_s$ represent, of course, the components of the vectors $\widehat{\boldsymbol{P}}$ and $\widehat{\boldsymbol{W}}$.

RIASSUNTO (*)

Si introducono due nuovi tensori del secondo ordine di correlazione dello spazio tempo $\mathscr{W}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau)$ e $\boldsymbol{H}_{ik}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau)$, che sono semplici combinazioni lineari dei tensori di correlazione discussi nella Parte I di questo studio ($\boldsymbol{x}_1$, $\boldsymbol{x}_2$ sono i vettori di posizione di due punti e τ un ritardo nel tempo). Questi nuovi tensori sono intimamente connessi a certe generalizzazioni della densità di energia (mediata rispetto al tempo) e del vettore del flusso di energia. Si derivano le equazioni differenziali, che sono soddisfatte da $\mathscr{W}_{jk}$ e $\boldsymbol{H}_{jk}$ nello spazio libero, e da esse vengono dedotte quattro nuove leggi di conservazione. Al limite per $\boldsymbol{x}_1 \to \boldsymbol{x}_2$, $\tau \to 0$, due di queste leggi si riducono alle leggi usuali (in forma mediata rispetto al tempo) per la conservazione dell'energia e della quantità di moto in un campo elettromagnetico. Le altre due leggi, a tale limite, si riducono semplicemente ad identità banali, cosichè esse non hanno alcun analogo nella schema della teoria usuale.

(*) *Traduzione a cura della Redazione.*

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A Theory of Hyperfragments.

I. – Binding Energies and Level Structure (*).

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Summary. — A model for the structure of hyperfragments is developed which is similar to the shell model for ordinary nuclei. The projection theorem relations for s and p shell hyperfragments are worked out and the case of two-particle interactions is dealt with in detail. The interaction parameters for s and p shell hyperfragments are deduced by a fit with the experimental data and the goodness of fit achieved on the model is satisfactory. An interesting inference is that $^4He_\Lambda$ has zero spin and hence the kaon is pseudoscalar. A chart of the hyperfragments including energy levels, stability and possible spins is presented. As a further application of the projection theorem an expression is derived for the magnetic moments of various hypernuclei. The present model is compared with previous work and some comments on the plausibility of the model are made.

1. – Introduction.

It is now generally accepted that the interactions of strange particles are charge-independent and that the Λ-hyperon is a charge singlet ([1-3]): consequently its interactions with the proton and with the neutron are identical. Further, from the conservation of angular momentum alone one is able to conclude that the Λ-hyperon has spin $\frac{1}{2}$ in view of the large asymmetry of

(*) Supported in part by the U. S. Atomic Energy Commission.

(1) T. Nakano and K. Nishijima: *Progr. Theor. Phys.*, **10**, 581 (1953); K. Nishijima: *Progr. Theor. Phys.*, **12**, 107 (1954).

(2) M. Gell-Mann: *Phys. Rev.*, **92**, 833 (1953); *Proc. of the Fifth Rochester Conf.* (New York, 1955); *Proc. of the Sixth Rochester Conf.* (New York, 1956), ch. VIII, p. 23.

(3) *Strange particle production and strange particle interaction*: see *Proc. of the Seventh Rochester Conf.* and *Annual Intern. Conf. on High Energy Phys. at CERN* (1958).

the pion emitted in the weak decay mode $\Lambda \to p + \pi^-$ [4]. Thus the structure of the Λ-hyperon-nucleon interaction is qualitatively similar to the nucleon-nucleon interaction, but of course there is no reason to expect that either the strength or range of the two interactions could be the same. In fact, since the « tail » of the nucleon-nucleon potential is due to the exchange of a single pion, such a tail would be absent in the Λ-hyperon-nucleon interaction. On the other hand, since for moderately large kinetic energies the strong reaction $\Lambda + \mathcal{N} \to \Sigma + \mathcal{N}$ can take place, the « potential » should really be considered as a matrix in these two channels; and this « exchange potential » has a one-pion tail and hence its effect cannot be forgotten. In addition, one also notices that the single K-meson exchange contribution has a range comparable to the « diagonal » two-pion exchange potential. All this would suggest that the detailed form of the potential is extremely complicated and that a qualitative study of the mutual interaction would require detailed information on the scattering parameters over a wide range of energies.

At the present time no such detailed experimental information is available. On the other hand there is one source of information of the Λ-hyperon-nucleon interaction and this is provided by the data on hyperfragments. Clearly such information would not give us the potential but only a suitable average of this potential over the bound state wave functions. The situation is parallel to the study of the average properties of nuclear forces from the binding energies and excited state spacings and spin-parity assignments for complex nuclei. There is a wealth of comparatively precise data on the binding energies of hyperfragments; even the existence of certain bound configurations (showing that their binding energies are positive) yields a certain amount of qualitative information on the average features of the hyperon-nucleon interaction.

The problem itself is not new: several investigators, amongst whom DALITZ deserves special mention, have studied the relation between the hyperon-nucleon interaction and the hyperfragment binding energies (and decay properties) [5-9]. But all of these attempts involved the use of explicit nuclear wave functions either obtained by a variational method or more often deduced from « structure studies » on the nucleus to which the hyperon is « added » to form the hyperfragments. The first method is in principle capable of arbitrary accuracy if the interaction were known exactly; in practice the calculations

[4] T. D. LEE and C. N. YANG: *Phys. Rev.*, **109**, 1755 (1958).

[5] S. IWAO: *Progr. Theor. Phys.*, **13**, 111 (1955); S. IWAO and Y. YAMAGUCHI: *Soryushiron Kenkyu* (mimeographed Circular in Japanese), **12**, 574 (1956).

[6] R. H. DALITZ and B. W. DOWNS: *Phys. Rev.*, **110**, 958 (1958).

[7] R. H. DALITZ and B. W. DOWNS: *Phys. Rev.*, **111**, 967 (1958).

[8] R. H. DALITZ and L. LIU: *Phys. Rev.*, **116**, 1312 (1959).

[9] R. H. DALITZ: *Ninth Annual Intern. Conf. on High-Energy Physics* (Kiev, 1959), unpublished.

are tedious and in any case the many-body interactions are not known exactly; the trial wave functions used tacitly assume a model except, perhaps, in the case of the simplest nuclei. The nuclear structure used in the study of hyperfragment binding neglects the dynamical correlations in the presence of the strongly interacting hyperon. Neither of these methods is applicable to the higher mass number hyperfragments and special models have to be invoked. Finally a study of the level structure for excited states by these methods is even more difficult and the approximations are more questionable in this context.

For ordinary nuclei one has the corresponding problem of binding energies and level structure; but here one has a working model of a complex nucleus, namely the *j-j* coupling shell model (or *L-S* coupling shell model or intermediate coupling shell model) with residual interparticle interactions. While the basic reasons for the validity of this model are not fully understood at the present time, the success of this model is beyond doubt. The basic physical picture is that of a sequence of one-particle states (which may be considered to correspond to the wave functions of a particle in a potential well) which are occupied by the various nucleons, the total wave function being properly antisymmetrized; and the essential physical approximation is that the true states of the nucleons are well-approximated by a « pure » configuration at least for the low-lying levels.

Motivated by the success of the nuclear shell model we have developed an alternative approach to the study of hypernuclei in which the system is represented in terms of the product of a set of one-particle wave functions suitably antisymmetrized with interparticle interactions also included. The quantitative predictions using this model are greatly facilitated by the use of the so-called « projection theorem » which enables the matrix elements of a complicated operator (like the Hamiltonian) to be expressed in terms of matrix elements of the same operator for certain few-nucleon configurations ([10]); the projection theorem is of course only a mathematical tool and in itself adds nothing to the physical model. We have made a systematic study of the binding energies, spins, level structure and decay characteristics; in this report we shall not discuss decay dynamics and we shall confine ourselves to the simple case of two-particle interactions only. We shall assume throughout the discussion that charge independence holds so that the Λ-n and Λ-p interactions are the same.

In the following section a brief outline of the projection theorem is given and the explicit formulae for the expectation values of the energy for various configurations in terms of the two-particle matrix elements are derived. The hypernuclei considered fall into three natural categories: 1) those with $A \leqslant 5$

([10]) S. Goldstein and I. Talmi: *Phys. Rev.*, **102**, 589 (1956); S. P. Pandya: *Phys. Rev.*, **103**, 986 (1956); **108**, 1312 (1957); S. Iwao: The theorem is generalized to both *L-S* and *j-j* schemes and applied to to some complex nuclei: *On the projection theorem for the energy levels and binding energies of complex nuclei* (to be published).

contain only nucleons in the $s_{\frac{1}{2}}$ shell and we shall call them the « s-shell hyperfragments »; 2) those with $6 \leqslant A \leqslant 13$ contain only nucleons in the $s_{\frac{1}{2}}$, $p_{\frac{3}{2}}$ shells: and 3) those with $14 \leqslant A \leqslant 17$ which include nucleons in the $s_{\frac{1}{2}}$, $p_{\frac{3}{2}}$, $p_{\frac{1}{2}}$ shells. Hypernuclei with larger atomic number have not been observed so far; in fact, at the time of writing this report there are no $p_{\frac{1}{2}}$-shell hyperfragments which have been identified ([11]). The hyperon itself is always in the $s_{\frac{1}{2}}$ « shell ». We shall find it advantageous to separate the discussion of the s-shell and p-shell hyperfragments. For the former group we shall find that there are no bound excited states. The interaction parameters are deduced in either case from the binding energy data and level structure is then predicted. The experimental data used as well as the predicted level schemes are tabulated. The magnetic moments of hypernuclei are derived in terms of nuclear magnetic moments and the anomalous magnetic moment of the Λ-hyperon as a further application of the theorem.

Of particular importance is the determination of the ground state spin of ${}^4\mathrm{He}_\Lambda$ since this enables one to determine the parity of the kaon. Our conclusion is that ${}^4\mathrm{He}_\Lambda$ has only one bound state of zero spin and that, consequently, the observation of the reaction $\mathrm{K}^- + {}^4\mathrm{He} \to {}^4\mathrm{He}_\Lambda + \pi^-$ permits us to assert that the kaon is pseudoscalar ([12]). A brief communication on the principal results of this part of our investigation has already been published ([13]).

2. – Formalism.

2·1. *General configurations.* – We want to discuss first the most general form of the projection theorem for the hyperfragments assuming two-body $\mathcal{N}$-Λ forces and to express the relations in a suitable form for the analysis of the Λ binding energies. For this purpose we will introduce quantities ϱ, μ and η for specifying the nucleons in shells ϱ, μ and the Λ-particles in a shell η, *i.e.* ϱ, μ and η correspond to $\varrho \equiv t_\varrho s_\varrho l_\varrho$, $\mu \equiv t_\mu s_\mu l_\mu$ and $\eta \equiv t_\eta s_\eta l_\eta$ in L-S scheme and j_ϱ, j_μ and j_η in j-j scheme. These symbols are short notations for the one-particle labels corresponding to the isotopic spin (T-space), intrinsic spin (S-space), and orbital angular momentum (L-space) respectively in L-S scheme; but in j-j scheme we can express the isotopic spin state by the synmetry of the angular momentum state and so we have no need to introduce the T-space in this scheme, *i.e.* one J-space, is enough for our purpose.

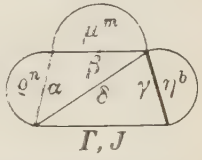
([11]) R. Ammar, R. Levi Setti, W. E. Slater, S. Limentani, P. E. Schlein and P. H. Steinberg: *Nuovo Cimento* (to be published).

([12]) M. M. Block, E. B. Brucker, I. S. Hughes, T. Kikuchi, C. Meltzer, F. Anderson, A. Pevsner, E. M. Harth, J. Leitner and H. O. Cohn: *Phys. Rev. Lett.*, **3**, 291 (1959).

([13]) S. Iwao and E. C. G. Sudarshan: *Phys. Rev. Lett.*, **4**, 140 (1960).

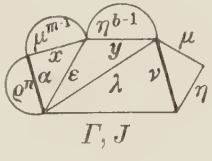
Now let us consider a general system with n nucleons in ϱ, m nucleons in μ and b Λ-particles in η and they couple to α, β, and γ respectively. The α and β couple to δ and then couple with γ to form the total Γ. It will be convenient to assign the total angular momentum J in addition to the short notation Γ because in L-S scheme the energy levels are degenerate with respect to the total angular momentum J. From our discussion it is clear that α, β, γ, δ and Γ are each the short notations for the corresponding three quantities (quantum numbers) $T\,S\,L$, in L-S scheme and the quantum numbers of angular momentum in j-j scheme. The state of hyperfragments is expressed by [14]

$$(1)\qquad \Psi_{\Gamma J}((\varrho)^n(\mu)^m;\,(\eta)^b) = \sum A(\alpha,\beta;\,\delta\,|\,\gamma)\,\varphi_{\Gamma J}(\alpha,\beta;\,\delta\,|\,\gamma) \equiv \sum A$$



where $A(\alpha,\beta;\,\delta\,|\,\gamma)$ is the suitable numerical coefficient and the round arc on the French diagram shows that the particles in it are antisymmetrized. The summations are on the possible quantum numbers, α, β, δ and γ. We will rewrite the French diagram in (1) in a form suitable to calculate the two-body $\mathcal{N}$-Λ interaction. To calculate the two-body matrix element of $\mathcal{N}$-Λ interaction for a nucleon in μ and a Λ-particle in η will we write

$$(2)\qquad \begin{cases} \varphi_{\Gamma J}(\alpha,\beta;\,\delta\,|\,\gamma) = \displaystyle\sum_{x,y,\varepsilon,\lambda,\nu} \langle \mu^m\beta\,|\,\mu^{m-1}x\rangle\langle\eta^b\gamma\,|\,\eta^{b-1}y\rangle \\ U(\alpha x\,\delta\mu;\,\varepsilon\beta)\,\overline{X}\begin{pmatrix} \varepsilon & \mu & \delta \\ y & \eta & \gamma \\ \gamma & \nu & \Gamma \end{pmatrix} \end{cases}$$



where the bracket $\langle A^c B\,|\,A^{c-1}C\rangle$, $U(abcd;\,ef)$ and $\overline{X}\begin{pmatrix} a & b & e \\ c & d & f \\ g & g & i \end{pmatrix}$ are respectively the coefficient of fractional parentage for c particles to $c-1$ and 1 particle, the Racah coefficient multiplied by some coefficient, *i.e.* $U(abcd;\,ef) = [ef]^{\frac{1}{2}}W(abcd;\,ef)$ and the X-coefficient multiplied by some coefficient;

$$\overline{X}\begin{pmatrix} a & b & e \\ c & d & f \\ g & h & i \end{pmatrix} = [efgh]^{\frac{1}{2}}X\begin{pmatrix} a & b & e \\ c & d & f \\ g & h & i \end{pmatrix}.$$

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[14] The notation used here is a natural extension of that of J. B. French for the ordinary nucleus. J. B. French: *The nuclear shell model* - I (March 1958) N7onr-32505. Lectures during the Fall Term (1957) at The University of Rochester.

Here we used the short notation $[abc\ldots] = (2a+1)(2b+1)(2c+1)\ldots$.

The same for a nucleon in ϱ and a Λ-particle in η is obtained by replacing $\mu \to \varrho$, $x \to z$, $\beta \to \alpha$, $\varepsilon \to \xi$, $\lambda \to \varkappa$, $\nu \to \omega$ and $m \to n$.

We are interested in $\mathcal{N}$-Λ interaction so we do not have to consider the $\mathcal{N}$-$\mathcal{N}$ interaction explicitly. The two-body $\mathcal{N}$-Λ interaction Hamiltonian of the system is given by

$$H = \sum H_{ij} = \sum_{\substack{\text{nucleons} \\ \text{in } \varrho}} H_{ij} + \sum_{\substack{\text{nucleons} \\ \text{in } \mu}} H_{ij}\,, \tag{3}$$

where we have separated out the two-body operator for nucleons in ϱ and in μ; i and j are suffixes for a nucleon and a Λ-particle respectively. The expectation value $E_{\Gamma J}(\alpha, \beta, \delta : \gamma | \alpha', \beta', \delta' : \gamma')$ of (3) with respect to (2) is given by

$$E_{\Gamma J}(\alpha, \beta, \delta : \gamma | \alpha', \beta', \delta' : \gamma) = mb \sum_{x,y,\varepsilon,\lambda,\nu} \langle \mu^m \beta | \mu^{m-1} x \rangle \langle \mu^m \beta' | \mu^{m-1} x \rangle \cdot \tag{4}$$

$$\cdot \langle \eta^b \gamma | \eta^{b-1} y \rangle \langle \eta^b \gamma' | \eta^{b-1} y \rangle U(\alpha x\, \delta\mu;\, \varepsilon\beta) U(\alpha' x\, \delta'\mu;\, \varepsilon\beta') \cdot$$

$$\cdot \bar{X} \begin{pmatrix} \varepsilon & \mu & \delta \\ y & \eta & \lambda \\ \lambda & \nu & \Gamma \end{pmatrix} \bar{X} \begin{pmatrix} \varepsilon & \mu & \delta' \\ y & \eta & \gamma' \\ \gamma & \nu & \Gamma \end{pmatrix} \cdot E_\nu(\mu, \eta) + (\text{the term which is obtained by re-}$$

placing $\mu \to \varrho$, $x \to z$, $\alpha \to \beta$, $\alpha' \to \beta'$, $\beta \to \alpha$, $\beta' \to \alpha'$, $\lambda \to \varkappa$, $\nu \to \omega$, $m \to n$),

where $E_A(e, f)$ is the elementary two-body matrix element of the nucleon in e and the Λ in f in a two particle state A. The coefficient A in (1) will be determined by solving the secular equation belonging to the same Γ and the different α, β, δ and γ for E (eq. (4)). It is always equal to unity for all the cases considered in this paper.

The nucleons in the hyperfragments with $A \leqslant 5$ are always in s-shell so there is no spin-orbit interaction. We will assume L-S coupling for the hyperfragments with $A \leqslant 5$ and j-j coupling for those with $A \leqslant 6$ because the Λ is always in the s-state. Under this assumption we will derive the expressions for the Λ binding energies of the hyperfragments in the following.

(Essentially, L-S and j-j results are completely the same for $j - \frac{1}{2}$ shell so the reader has no need to consider the L-S scheme. Compare Table I and Table III for $p_{\frac{1}{2}}$-shell hyperfragments.)

2·2. *Binding energy of the Λ-particle.* – The expression given in (4) looks complicated but it is simplified if we use the explicit form of the fractional parentage coefficients especially when they have a simple analytical form with respect to the summation indicated there.

The simple configurations are for a hole to two holes and a particle, two particles to a particle and a particle, and three particles to two particles and a particle. For these we have,

i) Closed shell to one hole and one particle:

$$\langle \varrho^N 0 \,|\, \varrho^{N-1}\varrho \rangle = (-)^{[\varrho]} \qquad \text{both for } L\text{-}S \text{ and } j\text{-}j \text{ scheme.}$$

ii) A hole to two holes and one particle:

$$\langle \varrho^{N-1}\varrho \,|\, \varrho^{N-2}\gamma \rangle = \sqrt{2/N(N-1)}\,(-)[\gamma]^{\frac{1}{2}} \qquad \text{for } L\text{-}S \text{ scheme}$$

$$\langle j^{N-1}j \,|\, j^{N-2}x \rangle^2 = ([x]/N(N-1))\cdot(1+(-)^x) \qquad \text{for } j\text{-}j \text{ scheme.}$$

iii) Two particles to one particle and one particle:

$$\langle \varrho^2 x \,|\, \varrho^1\varrho \rangle = 1 \qquad \text{both for } L\text{-}S \text{ and } j\text{-}j \text{ scheme.}$$

iv) Three particles to two particles and one particle:

$$\langle j^3 J \,|\, j^2 J' j \rangle^2 = (2j-3)\,\delta_{J'0}/3(2j-1) + 4\varepsilon_{J'}[J']/3[j](2j-1)$$

for seniority $v=1$ only in j-j scheme ([15]),

where $\varepsilon_{J'} = \frac{1}{2}(1+(-)^{J'})$.

All the hyperfragments which we are going to investigate are treated by i), ii), and iii), so we have no need to consider the last expression iv). To derive the explicit form of the binding energy of the Λ-particles we will summarize the notations used in this paper.

Multiplying a negative sign to the binding energy of the Λ-particle with the mass number A, $-B_\Lambda(A-1)$, we can write it as the sum of the expectation value of the one-particle Hamiltonian for the Λ-particle T_Λ and the expectation value of the potential energy between the Λ and the $A-1$ nucleons from the definition.

For the hyperfragments with $A \leqslant 5$ there are only two parameters (in addition to T_Λ) S_1, S_0 which are the triplet and singlet matrix elements of the Λ-nuclear interaction for a $s_{\frac{1}{2}}$ nucleon and a $s_{\frac{1}{2}}$ hyperon. It is convenient to consider the binding energy of the Λ in the hyperfragments with nucleons in $p_{\frac{3}{2}}$-shell by subtracting that of ${}^5\mathrm{He}_\Lambda$, B_Λ (4), because T_Λ is automatically subtracted and it is also convenient to subtract the Λ binding energy of ${}^{13}\mathrm{C}_\Lambda$ for the consideration of the hyperfragments with the nucleons in $p_{\frac{1}{2}}$-shell.

([15]) C. SCHWARTZ and A. DE-SHALIT: *Phys. Rev.*, **94**, 1257 (1954).

Then for these hyperfragments there appear four new parameters: two for the $J=1$ and 0 states from the coupling of a $p_{\frac{1}{2}}$ nucleon and $s_{\frac{1}{2}}$ hyperon; and two for $J=2$ and 1 states from the coupling of a $p_{\frac{3}{2}}$ nucleon and $s_{\frac{1}{2}}$ hyperon. We denote them by P_1, P_0, P_2', P_1' respectively. There is no observed hyperfragment belonging to the former category at this moment ([11]).

I) Hyperfragments with $A \leqslant 5$:

The binding energy of the Λ-particle is obtained as follows:

Two nucleons and one hyperon:

$$-B_\Lambda(2) = T_\Lambda + 2\sum_x [\alpha\alpha']^{\frac{1}{2}} W(\varrho\varrho I\eta;\alpha x)W(\varrho\varrho I\eta;\alpha' x)S_x\,, \tag{5}$$

where α and α' are the quantum numbers specifying two nucleon states.

Three nucleons and one hyperon:

$$-B_\Lambda(3) = T_\Lambda + \tfrac{1}{2}\sum_{x,\gamma'} [\varrho x\gamma](-)^{2(\varrho+x-\eta)} W(\varrho\varrho Ix;\gamma\eta)W(\varrho\varrho Ix;\gamma\eta)\,S_x\,, \tag{6}$$

where γ' specifies the two nucleon state which appeared after the transformation from one hole to two particles and one particle states.

Four nucleons and one hyperon:

$$-B_\Lambda(4) = T_\Lambda + 4/[\varrho I]\sum_x [x]\,S_x\,. \tag{7}$$

This become simple because of the simple form of the fractional parentage coefficient for the closed shell to one hole and one particle. It must be remarked that the formulas (5)–(7) are those in L-S scheme therefore all the summations have to be done on the three quantum numbers in $T\,S\,L$-space, *i.e.* the coefficients of S- are the triple products in the three spaces. The γ' sum in eq. (6) should be done only for those values of γ' which make the two nucleon states antisymmetric. Terms non-diagonal with respect to the states of the nucleon group coefficients vanish so we do not have to discriminate them explicitly. This means that the basic state (2) is itself the eigenstate of the isotopic spin. The same is true for the hyperfragments with the nucleons in p-shell. One must be careful if one wants to apply all the formulas derived in this paper to the hyperfragments with the excitation of the nucleon group ([10]) (where A in (1) is not unity.)

II) Hyperfragments with $6 \leqslant A \leqslant 13$:

In the following we will give the Λ binding energy in the form $-B_\Lambda(A-1)+B_\Lambda$ (4). We will use the j-j scheme for the nucleons since it is known that the observed nuclear spins of the ordinary nuclei with $5 \leqslant A \leqslant 16$ are

consistent with the value predicted from the shell model configurations [16]. Usually the nuclear levels in these regions are discussed using an intermediate coupling scheme [17]. But for the bound and low-lying excited states we will continue to use the j-j scheme and defer to a later communication the analysis in terms of the intermediate coupling scheme. We are interested in the lower bound states of the nucleons and the Λ-particle; so we assume the spins of the nucleon group are not different from those in their ground state which then couple to the Λ spin $\frac{1}{2}$. We will discuss the two possibilities of the anti-parallel and parallel orientations of the spin of the nucleon group and the Λ spin. This is another reason for the choice of the coupling scheme. We will give the formula for the binding energy of the hyperfragments for each mass number in the following.

(8) $^{6}\mathrm{He}_{\Lambda}$ and $^{6}\mathrm{Li}_{\Lambda}$:
$$-B_{\Lambda}(5)+B_{\Lambda}(4)=P'_{x}\delta_{xJ},$$

(9) $^{7}\mathrm{He}_{\Lambda}$ and $^{7}\mathrm{Li}_{\Lambda}$:
$$-B_{\Lambda}(6)+B_{\Lambda}(4)=$$
$$=2[J_{\alpha}J_{\alpha'}]^{\frac{1}{2}}\sum_{x}[x]W(jjJj';J_{\alpha}x)W(jjJj';J_{\alpha'}x)P'_{x},$$

(10) $^{8}\mathrm{Li}_{\Lambda}$ and $^{8}\mathrm{Be}_{\Lambda}$:
$$-B_{\Lambda}(7)+B_{\Lambda}(4)=P'_{x}\delta_{xJ}+2/[jj']\sum_{x}[x]P'_{x},$$

(11) $^{9}\mathrm{Be}_{\Lambda}$:
$$-B_{\Lambda}(8)+B_{\Lambda}(4)=(4/[jj'])\sum_{x}[x]P'_{x},$$

(12) $^{9}\mathrm{Li}_{\Lambda}$:
$$-B_{\Lambda}(8)+B_{\Lambda}(4)=5\sum_{x}[x](W(jjJj';2x))^{2}P'_{x}-$$
$$-(1/[j'])\sum_{x}[x]P'_{x}+5\sum_{x,y}(-)^{x+y}[xy](W(jy2j';Jj))^{2}W(jyxj;j'j')P'_{x},$$

(13) $^{10}\mathrm{Be}_{\Lambda}$ and $^{10}\mathrm{B}_{\Lambda}$:
$$-B_{\Lambda}(9)+B_{\Lambda}(4)=(2/[jj'])\sum_{x}[x]P'_{x}+$$
$$+(1/[j'])\sum_{x}[x]P'_{x}-\sum_{x}[x]W(jj'j'j;Jx)P'_{x},$$

(14) $^{11}\mathrm{B}_{\Lambda}$:
$$-B_{\Lambda}(10)+B_{\Lambda}(4)=(2/[j'])\sum_{x}[x]P'_{x}+$$
$$+14\sum_{x,y}(-)^{x+y}[yx](W(jj3j';Jj))^{2}W(jyxj;j'j')P'_{x},$$

(15) $^{12}\mathrm{B}_{\Lambda}$ and $^{12}\mathrm{C}_{\Lambda}$:
$$-B_{\Lambda}(11)+B_{\Lambda}(4)=(1/[j'])\sum_{x}[x]P'_{x}-$$
$$-\sum_{x}[x]W(jj'j'j;Jx)P'_{x}+(4/[jj'])\sum_{x}[x]P'_{x},$$

(16) $^{13}C_{\Lambda}$:
$$-B_{\Lambda}(12)+B_{\Lambda}(4)=(8/[jj'])\sum_{x}[x]P'_{x},$$

where j and j' are for nucleons and Λ-particle respectively.

[16] M. G. Mayer: *Phys. Rev.*, **74**, 235 (1948); **78**, 16 (1950).
[17] J. P. Elliott and A. M. Lane: *Handb. d. Phys.*, vol. **39** (Berlin, 1957).

III) Hyperfragments with $14 \leqslant A \leqslant 17$:

No experimental data are available for these hyperfragments; but it is important to have the experimental knowledge for these because they give the knowledge on the phenomenological $\mathcal{N}$-Λ forces.

The derivation and the results for the Λ binding energies are simpler than those in II). They are given by

(17) $^{14}C_\Lambda$ and $^{14}N_\Lambda$: $$-B_\Lambda(13)+B_\Lambda(12)=P_x\delta_{xJ},$$

(18) $^{15}N_\Lambda$: $$-B_\Lambda(14)+B_\Lambda(12)=$$
$$=2\sum_x[J_\alpha J_{\alpha'}]^{\frac{1}{2}}[x]W(jjJj';J_\alpha x)W(jjJj';J_{\alpha'}x)P_x,$$

(19) $^{16}N_\Lambda$ and $^{16}O_\Lambda$: $$-B_\Lambda(15)+B_\Lambda(12)=P_x\delta_{xJ}+(2/[jj'])\sum_x[x]P_x,$$

(20) $^{17}O_\Lambda$: $$-B_\Lambda(16)+B_\Lambda(12)=(4/[jj'])\sum_x[x]P_x.$$

It is needless to say that (17)-(20) apply to the hyperfragments with $2s$ shell nucleons only by the modification of the left hand side by replacement of the corresponding « $2s$-shell hyperfragment » binding energies.

3. – Interaction parameters S_0 and S_1.

In this section we want to determine the interaction parameters S_1 and S_0 from the binding energies of the hyperfragments with the mass number $A=3$, 4 and 5. In order to obtain the results we will put the values

TABLE I. – *Expectation value of the two-body $\mathcal{N}$-Λ forces for the hyperfragments with $A=3$, 4, and 5.*

Hyperfragment	J	$-B_\Lambda(A-1)$	Coefficients of		
			T_Λ	S_0	S_1
$^3H_\Lambda$	1/2	$-B_\Lambda(2)$	1	3/2	1/2
	3/2	$-B_\Lambda(2)$	1	0	2
$^4H_\Lambda$ and $^4He_\Lambda$	0	$-B_\Lambda(3)$	1	3/2	3/2
	1	$-B_\Lambda(3)$	1	1/2	5/2
$^5He_\Lambda$	1/2	$-B_\Lambda(4)$	1	1	3

$t_\varrho s_\varrho l_\varrho = \frac{1}{2}\,\frac{1}{2}\,0$ for the nucleon and $t_\eta s_\eta l_\eta = 0\,\frac{1}{2}\,0$ for the Λ-particle. The coupling between nucleons in ${}^3H_\Lambda$, ${}^4H_\Lambda$ (${}^4He_\Lambda$) and ${}^5He_\Lambda$ are respectively $T_\Lambda S_\Lambda L_\Lambda = 0\,1\,0$, $\frac{1}{2}\,\frac{1}{2}\,0$ and $0\,0\,0$. Therefore the total spins of these hyperfragments are either $\frac{1}{2}$, 0, $\frac{1}{2}$ or $\frac{3}{2}$, 1, $\frac{1}{2}$ corresponding to the antiparallel and parallel spins of the nucleons and hyperon.

Putting these values in the formulas (5), (6), and (7) we obtain the results shown in Table I.

The coefficients obtained in Table I for S_0 and S_1 are the same as those of DALITZ *et al.* ([9]).

The experimental values of the Λ binding energies for $A = 3$, 4 and 5 are given in Table II. From the charge independence of the strong interaction there is no reason for the Λ binding energies of ${}^4H_\Lambda$ and ${}^4He_\Lambda$ to be different. They coincide with each other within the experimental error ([11]), but we will choose first one and then the other for determining the parameters, (*a*) and (*b*), as shown in Table II.

TABLE II. – Λ *binding energies of the hyperfragments with* $A = 3$, 4, *and* 5 (*in* MeV).

Set (*a*)	${}^3H_\Lambda$	0.12	${}^4H_\Lambda$	2.20	${}^5He_\Lambda$	3.08
Set (*b*)	${}^3H_\Lambda$	0.12	${}^4He_\Lambda$	2.36	${}^5He_\Lambda$	3.08

Combining Tables I and II we obtain for the «antiparallel spin favoured» case

$$(21)\qquad \begin{cases} T_\Lambda + \frac{3}{2}S_0 + \frac{1}{2}S_1 = -.12 \text{ MeV}, \\ T_\Lambda + \frac{3}{2}S_0 + \frac{3}{2}S_1 = -2.20 \quad \text{or} \quad -2.36 \text{ MeV}, \\ T_\Lambda + S_0 + 3S_1 = -3.08 \text{ MeV}. \end{cases}$$

From (21) we get,

(21.*a*) $T_\Lambda = 7.64$ MeV, $S_0 = -4.46$ MeV, $S_1 = -2.08$ MeV, for set (*a*) in Table II.

(21.*b*) $T_\Lambda = 8.92$ MeV, $S_0 = -5.28$ MeV, $S_1 = -2.25$ MeV, for set (*b*) in Table II.

It is seen that the slight change of the binding energy of the $A=4$ hyperfragment has a big effect on the values deduced for T_Λ and S_0. For the « parallel spin favoured » case

$$(22)\qquad \begin{cases} T_\Lambda + \qquad 2S_1 = -.12 \text{ MeV}, \\ T_\Lambda + \tfrac{1}{2}S_0 + \tfrac{5}{2}S_1 = -2.20 \quad \text{and} \quad 2.36 \text{ MeV}, \\ T_\Lambda + \quad S_0 + 3S_1 = -3.08 \text{ MeV}. \end{cases}$$

The determinant of the coefficients of T_Λ, S_0, and S_1 vanishes and so formula (22) gives an inconsistent set mathematically. Subtracting the first equation from the second and third we have

$$(22')\qquad \begin{cases} \tfrac{1}{2}S_0 + \tfrac{1}{2}S_1 = -2.2 \text{ MeV}, \\ S_0 + \quad S_1 = -3.0 \text{ MeV}. \end{cases}$$

These are inconsistent by several times the standard deviation of an experimental error (*).

As seen from (21.*a*) and (21.*b*) both singlet and triplet interactions are attractive.

4. – KΛ relative parity and some properties of hyperfragments with $A \leqslant 5$.

It is possible to estimate the one-particle energy (the « kinetic energy ») of the Λ-particle in the hyperfragments from the momentum distribution of the Λ obtained from the 160 π-mesic decays of the hyperfragments ${}^{15}\text{H}{\cdot}\text{e}_\Lambda$ ([18]) which gives about (120÷140) MeV/c assuming the Maxwell distribution of the Λ momentum. The corresponding « kinetic energy » of the Λ-particle is (6.5÷8.6) MeV which is consistent with the values given in (21) and (22).

We can calculate the energy levels of the « parallel spin » of $A=3$ and 4, $J=\frac{3}{2}$ and 1, by combining Table I with (21) and (22). They are 3.60 MeV and 2.40 MeV for the set (21) and 4.6 MeV and 3.04 MeV for the set (22) respectively. Therefore both the « parallel spin » states are unbound (by 3.48 MeV and 0.20 MeV for the former and 4.44 MeV and 0.68 MeV for the

(*) This was pointed out by Prof. M. M. Block; the author expresses his thanks to him on this point.

([18]) R. Ammar, R. Levi-Setti, W. E. Slater, S. Limentani, P. E. Schlein and P. H. Steinberg: *Nuovo Cimento*, **13**, 1156 (1959).

latter). The values 0.20 MeV and 0.68 MeV are large compared to the experimental error on the binding energy determination of $^3He_\Lambda$, 0.12 MeV. The only bound state of $^4He_\Lambda$ has spin 0 and consequently the existence of the mesic absorption of the negative kaon in helium ([12]),

$$K^- + {}^4He \to {}^4He_\Lambda + \pi^- \tag{23}$$

implies that the kaon parity (relative to that of the Λ-hyperon) is odd, when the spin of the kaon is zero ([19]) and the parity is conserved in the reaction process; hence the KΛ relative parity should be odd.

It must be emphasized that this conclusion does not depend on the relative angular momentum of K and ^{4}He. There is no ambiguity as encountered in the K^--p and K^--d processes depending upon whether the kaon is absorbed either from s state or p state ([20-22]).

It is easily seen that pΛ and nΛ bound states are not allowed. The nonexistence of the bound $^3He_\Lambda$ and nnΛ is proved as follows:

Putting $T_\Lambda S_\alpha L_\alpha = 1\ 0\ 0$, $T_\Gamma S_\Gamma L_\Gamma = 1\ \frac{1}{2}\ 0$ in (5) we obtain,

$$-B_\Lambda(2) = T_\Lambda + \tfrac{1}{2}S_0 + \tfrac{3}{2}S_1\,. \tag{24}$$

The set of values (21) and (22) gives $B_\Lambda(2) = -2.28$ MeV and -2.92 MeV respectively. So there is no bound state of $^3He_\Lambda$ and nnΛ. The demonstration given here does not depend on the form of the $\mathcal{N}$-Λ potential. So it is more general than the method which uses the wave depth parameter ([5]).

The expectation values of the tensor and the spin-orbit interaction are zero for the S state. If we assume the same radial dependence for the central spin-independent and spin-dependent $\mathcal{N}$-Λ interaction

$$V_{\mathcal{N}\Lambda}(r) = (1 + a\boldsymbol{\sigma}_{\mathcal{N}}\cdot\boldsymbol{\sigma}_\Lambda)V(r) \tag{25}$$

can determine the coefficient a from the equation

$$\frac{1-3a}{1+a} = \frac{S_0}{S_1}\,. \tag{26}$$

Putting the values S_0, S_1 given in (21), (22) we have $a = -.22$ and $-.25$ for the sets, (a) and (b) of Table II respectively.

([19]) R. H. Dalitz: *Rep. Progr. Phys.*, **20**, 163 (1957).

([20]) T. B. Day, G. A. Snow and J. Sucher: *Phys. Rev. Lett.*, **3**, 61 (1959).

([21]) R. K. Adair: *Phys. Rev. Lett.*, **3**, 438 (1959).

([22]) J. Leitner, P. Nordin, A. H. Rosenfeld, F. T. Solmitz and R. D. Tripp: *Phys. Rev. Lett.*, **3**, 238 (1959).

5. – Interaction parameters P_1' and P_2' and P_0, P_1.

The parameters P_1', P_2' and P_0, P_1 are derived from the binding energies of the hyperfragments with $6 \leqslant A \leqslant 13$ and $14 \leqslant A \leqslant 17$ respectively.

Unfortunately we do not have any data for the latter group ([11]) but it may be observed in the future so we will derive the numerical coefficients for both groups. The binding energies of hyperfragments with $6 \leqslant A \leqslant 13$ can be obtained from (6)–(16) putting $j = \frac{3}{2}$, $j' = \frac{1}{2}$ and the observed spin of the nucleons J ([24]). The same for $14 \leqslant A \leqslant 17$ can be obtained from (17)–(20) putting $j = \frac{1}{2}$ instead of $\frac{3}{2}$.

The assumption made here for the spins of nucleons give the possible total spin of the hyperfragment as shown in Table III. All the coefficients derived here satisfy the complete charge independence requirement, *i.e.* the non-diagonal term with respect to a different seniority of the nucleons vanishes automatically ([25]).

Now we come to the determination of the parameters P_1', P_2' from the experimental data.

I) ${}^{7}\mathrm{Li}_{\Lambda}$, ${}^{8}\mathrm{Li}_{\Lambda}$, ${}^{9}\mathrm{Be}_{\Lambda}$, and ${}^{9}\mathrm{Li}_{\Lambda}$.

The experimental values for the binding energies of these hyperfragments are more reliable than those of other hyperfragments because of the former we know 15, 8, 6, and 4 events and of the others in the range $6 \leqslant A \leqslant 13$ we know only one event, except ${}^{12}\mathrm{B}_{\Lambda}$ ([23]). ${}^{7}\mathrm{Li}_{\Lambda}$ decay is classified into two-body mesic and other mesic decays; 10 events belong to the former and 5 events belong to the latter. The Λ binding energies obtained from both groups differ by 0.50 MeV. AMMAR *et al.* ([11]) considered that this difference comes from the fact that the two-body mesic mode occurs through the decay into a pion and the first excited state of the ${}^{7}\mathrm{Be}$ which is 0.43 MeV above the ground state of ${}^{7}\mathrm{Be}$. Our theorem can test their assumption and it is shown that it is quite plausible (see the following discussion).

Combining the four pieces of experimental data discussed above with the coefficients given in Table III we obtain the following values by a least squares fit (see Table IV):

$$P_1' = -1.41 \text{ MeV}, \qquad P_2' = -0.551 \text{ MeV} \tag{27}$$

([23]) Two new events, one for ${}^{12}\mathrm{B}_{\Lambda}$ and one for ${}^{13}\mathrm{C}_{\Lambda}$ were reported by SCHLEIN and SLATER: *Bull. Amer. Phys. Soc.*, **5**, 12 (1960). The first of these values is obtained by taking the average with the former single event ([11]).

([24]) F. AJZENBERG-SELOVE and T. LAURITSEN: *Nucl. Phys.*, **11**, 1 (1959).

([25]) See S. IWAO: ref. ([10]).

TABLE III. – *Expectation values of the two-body $\mathcal{N}$-Λ forces for the hyperfragments with $6 \leqslant A \leqslant 13$ and $14 \leqslant A \leqslant 17$.*

Hyperfragment	J	$-B_\Lambda(A-1)+B_\Lambda(4)$	Coefficients of	
			P'_1	P'_2
$^6He_\Lambda$ and $^6Li_\Lambda$	1	$-B_\Lambda(5)+B_\Lambda(4)$	1	0
$^6He_\Lambda$ and $^6Li_\Lambda$	2	$-B_\Lambda(5)+B_\Lambda(4)$	0	1
$^7He_\Lambda$	1/2	$-B_\Lambda(6)+B_\Lambda(4)$	3/4	5/4
$^7Li_\Lambda$	1/2	$-B_\Lambda(6)+B_\Lambda(4)$	5/4	3/4
$^7Li_\Lambda$	3/2	$-B_\Lambda(6)+B_\Lambda(4)$	1/2	3/2
$^8Li_\Lambda$ and $^8Be_\Lambda$	1	$-B_\Lambda(7)+B_\Lambda(4)$	7/4	5/4
$^8Li_\Lambda$ and $^8Be_\Lambda$	2	$-B_\Lambda(7)+B_\Lambda(4)$	3/4	9/4
$^9Be_\Lambda$	1/2	$-B_\Lambda(8)+B_\Lambda(4)$	3/2	5/2
$^9Li_\Lambda$	3/2	$-B_\Lambda(8)+B_\Lambda(4)$	9/4	7/4
$^9Li_\Lambda$	5/2	$-B_\Lambda(8)+B_\Lambda(4)$	1	3
$^{10}Be_\Lambda$ and $^{10}B_\Lambda$	1	$-B_\Lambda(9)+B_\Lambda(4)$	5/2	5/2
$^{10}Be_\Lambda$ and $^{10}B_\Lambda$	2	$-B_\Lambda(9)+B_\Lambda(4)$	3/2	7/2
$^{11}B_\Lambda$	5/2	$-B_\Lambda(10)+B_\Lambda(4)$	13/2	11/4
$^{11}B_\Lambda$	7/2	$-B_\Lambda(10)+B_\Lambda(4)$	3/2	9/2
$^{12}B_\Lambda$	1	$-B_\Lambda(11)+B_\Lambda(4)$	13/4	15/4
$^{12}B_\Lambda$	2	$-B_\Lambda(11)+B_\Lambda(4)$	9/4	19/4
$^{13}C_\Lambda$	1/2	$-B_\Lambda(12)+B_\Lambda(4)$	3	5
Hyperfragment	J	$-B_\Lambda(A-1)+B_\Lambda(12)$	P_0	P_1
$^{14}C_\Lambda$ and $^{14}N_\Lambda$	0	$-B_\Lambda(13)+B_\Lambda(12)$	1	0
$^{14}C_\Lambda$ and $^{14}N_\Lambda$	1	$-B_\Lambda(13)+B_\Lambda(12)$	0	1
$^{15}N_\Lambda$	1/2	$-B_\Lambda(14)+B_\Lambda(12)$	3/2	1/2
$^{15}N_\Lambda$	3/2	$-B_\Lambda(14)+B_\Lambda(12)$	0	2
$^{16}N_\Lambda$ and $^{16}O_\Lambda$	0	$-B_\Lambda(15)+B_\Lambda(12)$	3/2	3/2
$^{16}N_\Lambda$ and $^{16}O_\Lambda$	1	$-B_\Lambda(15)+B_\Lambda(12)$	1/2	5/2
$^{17}O_\Lambda$	1/2	$-B_\Lambda(16)+B_\Lambda(12)$	1	3

for « antiparallel spins », and

$$P_1' = -0.051 \text{ MeV}, \qquad P_2' = -1.38 \text{ MeV} \tag{28}$$

for « parallel spins ». Both $J=1$ and 2 states give attractive interactions: the $J=1$ state in the former is much more attractive than the $J=2$ state but in the latter the contribution of the $J=1$ states is about 4% of that of the $J=2$ state.

In Table IV we compare the experimental and theoretical values. The least deviation $\left[\sum_i \Delta_i^2/(N-k)\right]^{\frac{1}{2}}$ (where Δ_i is the difference between the experimental and theoretical binding energies and N, k are respectively the number of data and that of parameters used for the fit) shows that the parameters explain the experimental data to within 5.1% and 6.6% for « antiparallel spins » and « parallel spins » respectively. As seen from Table IV, Δ_i for $^7\text{Li}_\Lambda$ are both negative for both the couplings of the spins; we will have a worse fit to the experiment if we use the data obtained from the two-body mesic decay (10 events). We found that even if we use the mean value of the binding energies of $^7\text{Li}_\Lambda$ obtained from all the 15 events we have 9% and 11% fit for the « antiparallel spins » and « parallel spins » respectively. Therefore we can conclude that the assumption made by AMMAR *et al.* for the analysis of $^7\text{Li}_\Lambda$ decay is reasonable ([11]) and consistent with our theory. There remains the question of how some strong selection rules forbid the decay into a pion and the ground state of ^{7}Be.

TABLE IV. – *Comparison between the experimental and the theoretical binding energies of hyperfragments with $P_{\frac{3}{2}}$ shell nucleons.*

Hyper-fragment	$-B_\Lambda(A-1)+B_\Lambda(4)$ in MeV (exp.)	$-B_\Lambda(A-1)+B_\Lambda(4)$ in MeV (theor.) for antiparallel spins	Δ_i	$-B_\Lambda(A-1)+B_\Lambda(4)$ in MeV (theor.) for parallel spins	Δ_i
$^7\text{Li}_\Lambda$	−2.38	−2.18	−.20	−2.10	−.28
$^8\text{Li}_\Lambda$	−3.03	−3.16	.13	−3.14	.11
$^9\text{Be}_\Lambda$	−3.52	−3.49	−.03	−3.53	.01
$^9\text{Li}_\Lambda$	−4.12	−4.14	.02	−4.19	.70

II) $^{7}He_{\Lambda}$, $^{8}Be_{\Lambda}$, $^{11}B_{\Lambda}$, $^{12}B_{\Lambda}$, $^{13}C_{\Lambda}$.

All the five hyperfragments given here are rare events and their Λ binding energies are $(3.0 \pm .7)$, $(6.6 \pm .6)$, $(9.9 \pm .6)$ ([11]), $(9.75 \pm .6)$ and $(10.8 \pm .6$ MeV ([23]) respectively. Combining Table III, (27), (28) and adding the Λ binding energy of $^{5}He_{\Lambda}$ we have theoretically 4.73, 6.24, 9.18, 9.73, 10.1 MeV and 4.84, 6.22, 9.37, 9.75, 10.1 MeV for the « antiparallel » and « parallel » spins respectively. As seen from these numbers they can be predicted well except the Λ binding energy of $^{7}He_{\Lambda}$.

6. – Hyperfragments with $A = 6$, 10 and 13: some « unobserved » hyperfragments.

I) Hyperfragments with $A = 6$ and 10.

Hyperfragments with mass number 6, 10, and 13 are not observet yet. All these hyperfragments belong to a group discussed in Section 5. The Λ binding energies of these hyperfragments can be calculated from Table III and (27), (28). Assuming the charge symmetry of $\mathcal{N}$-Λ forces we have 4.49, 7.98 MeV, and 4.46, 7.99 MeV for the « antiparallel » and « parallel » spins of $^{6}He_{\Lambda}$ ($^{6}Li_{\Lambda}$) and $^{10}Be_{\Lambda}$ ($^{10}B_{\Lambda}$). The binding energies (due to the $\mathcal{N}$-Λ forces) of the last proton and the neutron of $^{6}He_{\Lambda}$ and $^{6}Li_{\Lambda}$ are 1.41 and 1.38 MeV respectively and the binding energies of the last proton and neutron of the ^{5}Li and ^{5}He nuclei are -1.796 and $-.957$ MeV ([24]) respectively; so the $\mathcal{N}$-Λ force is not enough to bind the last proton of the $^{6}Li_{\Lambda}$ hyperfragment but the $^{6}He_{\Lambda}$ is a stable one.

II) Some comments on the charge symmetry of $\mathcal{N}$-Λ forces.

We would like to point out that we can predict the existence and non-existence of the unobserved hyperfragments from the charge symmetry hypothesis of $\mathcal{N}$-Λ forces (charge symmetry, and not charge independence). Since the early discussion of this point by DALITZ ([26]) many bound states of the Λ hyperon in nuclei have been identified and their binding energies have been determined. The observed binding energies of the Λ-particle of $^{4}H_{\Lambda}$ and $^{4}He_{\Lambda}$ coincide with each other within experimental error ([11]). The same is true for $^{8}Li_{\Lambda}$ and $^{8}Be_{\Lambda}$ ([11]) hyperfragments. We will proceed assuming the charge symmetry of $\mathcal{N}$-Λ forces from now on. The stability of the hyperfragments is not only determined from the binding energies of the Λ-particle but also from the stability of the nuclei in which the Λ-particles are bound. If the separation of a hyperfragment X_{Λ} into a hyperfragment Y_{Λ} and a

([26]) R. H. DALITZ: *Phys. Rev.*, **99**, 1475 (1955).

single nucleon or a nucleus e, or into a nucleus U and a Λ-particle

$$X_\Lambda \to Y_\Lambda + e, \tag{29}$$

$$\to U + \Lambda, \tag{30}$$

have a sufficiently larger lifetime than the free Λ-particle we can observe the bound state of X_Λ. To prevent us from observing the bound state, the interaction (29), (30) must either be a strong interaction or an allowed electromagnetic interaction; since we are considering the ground state of the hyperfragment, we can confine ourselves to the strong interaction. The charge symmetry conjugates of $^7He_\Lambda$, $^9Li_\Lambda$ and $^{12}B_\Lambda$ are $^7Be_\Lambda$, $^9B_\Lambda$ and $^{12}C_\Lambda$ respectively. To find out the possibility of the bound state of the hyperfragment X_Λ we have to examine the smallest separation energy for the processes (29) and (30). It will be done by examining the properties of a bound or an unbound state of the ordinary nucleus defined by (30). The most important contribution to this problem is the difference of the binding energies of the mirror hyperfragments due to the Coulomb effect. To find out the Coulomb energy difference of $^7He_\Lambda$ and $^7Be_\Lambda$ we have to use the difference of the experimental binding energies of ^{6}He and ^{6}Be nuclei. Unfortunately there is no bound state of ^{6}Be so we have to solve this in the following way. ^{5}He and ^{5}Li binding energies are known to be 27.34 and 26.50 MeV respectively ([27]). The sum of the differences of the binding energies of the above two groups of mirror nuclei gives us the Coulomb energy difference of $^7He_\Lambda$ and $^7Be_\Lambda$ hyperfragments. The value obtained in this way is 2.49 MeV ([27]). From the charge independence of the nuclear forces we can suppose that ^{6}Be is unbound by 1.542 MeV (*) which is bigger than the energy supplied by a Λ binding $(4.8 - 3.08)/2 - 0.86$ MeV for a proton in $P_{\frac{3}{2}}$ shell so that the total binding energy of the two $P_{\frac{3}{2}}$ protons in $^7Be_\Lambda$ is almost zero. (If we use the experimental binding energy (« one-event ») of $^7He_\Lambda$ we do not have a bound $^7Be_\Lambda$.) We feel that $^7He_\Lambda$ binding energy redetermination and a search for $^7Be_\Lambda$ bound states would be very interesting.

The binding energy difference of ^{8}Li and ^{8}B nuclei gives the Coulomb energy difference 3.171 MeV between $^9Li_\Lambda$ and $^9B_\Lambda$. The last proton separation energy of $^8B_\Lambda$ is $-.138$ MeV ([27]). Supplying the energy due to $\mathcal{N}$-Λ interaction $(7.8 - 308)/5 = .82$ MeV to the p-shell proton of $^9B_\Lambda$ we have 1.0 MeV binding for the last proton of $^9B_\Lambda$.

(*) According to a recent experiment ^{6}Be is unbound with respect to (^{4}He-2p) by 1.35 MeV. F. Ajzenberg-Selove, C. F. Osgood and C. P. Baker: *Phys. Rev.*, **116**. 1521 (1959).

([27]) A. H. Wapstra: *Physica*, **21**, 367 (1955).

(In the same way we can predict the existence of the bound state of $^{13}C_\Lambda$ which was really observed after our prediction (see (23)).)

7. – Energy levels and the chart of hyperfragments with $6 \leqslant A \leqslant 13$.

7·1. *Energy levels of hyperfragments.* – We will tabulate the theoretical prediction for the first excited states of the hyperfragments due to the « antiparallel » and « parallel » spins in Table V.

E_x in Table V is the excitation energy. It is seen from the Table that the level spacing for the parallel spins (in the ground states) is bigger than that for the antiparallel spins; and the level spacings between $J-1$ and 2 states remain constant with respect to the change of mass numbers.

TABLE V. – *Levels of hyperfragments with* $6 \leqslant A \leqslant 13$.

Hyperfragment	Antiparallel spins for the ground states		Parallel spins for the ground states	
	J	E_x (MeV)	J	E_x (MeV)
$^6He_\Lambda$ and $^6Li_\Lambda$	2	0.86	1	1.33
$^7Li_\Lambda$	3/2	0.65	1/2	1.00
$^8Li_\Lambda$ and $^8Be_\Lambda$	2	0.86	1	1.33
$^9Li_\Lambda$ and $^9B_\Lambda$	5/2	1.08	3/2	1.66
$^{10}Be_\Lambda$ and $^{10}B_\Lambda$	2	0.86	1	1.33
$^{11}B_\Lambda$	7/2	1.51	5/2	2.30
$^{12}B_\Lambda$ and $^{12}C_\Lambda$	2	0.86	1	1.33

7·2. *Chart of hyperfragments.* – It will be convenient to make a chart of hyperfragments predicted from our theory. We would like to list the stability, isotopic spin, and spin expected from our theory together with experimental knowledge for hyperfragments with $6 \leqslant A \leqslant 13$.

TABLE VI. – *Chart of hyperfragments.*

Mass no. A	T 1 / T_z −1	1/2 / −1/2	0 / 0	1/2 / 1/2	1 / 1
1			Λ s 1/2		
2		nΛ u 0, 1		pΛ u 0, 1	
3	nnΛ u 1/2		$^{3}H_{\Lambda}$ s 1/2		$^{3}He_{\Lambda}$ u 1/2
4		$^{4}H_{\Lambda}$ s 0		$^{4}He_{\Lambda}$ s 0	
5			$^{5}He_{\Lambda}$ s 1/2		
6		$^{6}He_{\Lambda}$ s 1, 2		$^{6}Li_{\Lambda}$ u 1, 2	
7	$^{7}He_{\Lambda}$ s 1/2		$^{7}Li_{\Lambda}$ s 1/2, 3/2		$^{7}Be_{\Lambda}$ s (1) 1/2
8		$^{8}Li_{\Lambda}$ s 1, 2		$^{8}Be_{\Lambda}$ s 1, 2	
9	$^{9}Li_{\Lambda}$ s 3/2, 5/2		$^{9}Be_{\Lambda}$ s 1,2		$^{9}B_{\Lambda}$ s 3/2, 5/2
10		$^{10}Be_{\Lambda}$ s 1/2		$^{10}B_{\Lambda}$ s 1, 2	
11	$^{11}Be_{\Lambda}$ s 1/2		$^{11}B_{\Lambda}$ s 5/2, 7/2		$^{11}C_{\Lambda}$ s 1/2
12		$^{12}B_{\Lambda}$ s 1, 2		$^{12}C_{\Lambda}$ s 1, 2	
13			$^{13}C_{\Lambda}$ s 1/2		

s = stable - u = unstable (for strong interaction)
The numbers after *s* and *u* show the possible spins of the lowest states.

8. – Magnetic moments of hyperfragments.

We will derive the projection theorem of magnetic moment of hyperfragments in this section. The operator of the magnetic moment for the hyperfragment is given by,

$$\boldsymbol{M} = \sum_i \boldsymbol{M}_i + \boldsymbol{M}_\Lambda , \tag{31}$$

where $\boldsymbol{M}_i$ and $\boldsymbol{M}_\Lambda$ are the operator of the magnetic moment of the i-th nucleon and that of the Λ-particle respectively. The operator has the vector transformation property. Then the total magnetic moment of the hyperfragment in L-S scheme is given as a function of the magnetic moment of $A-1$ nu-

TABLE VII. – *Magnetic moments of hyperfragments.*

Mass number A	Hyperfragment	J	Coefficients of	
			$\mu\,(A-1)$	μ_Λ
1	Λ	1/2	0	1
2	nΛ and pΛ	1	1	1
3	$^3H_\Lambda$	1/2	2/3	−1/3
3	$^3H_\Lambda$	3/2	1	1
4	$^4H_\Lambda$ and $^4He_\Lambda$	1	1	1
5	$^5He_\Lambda$	1/2	0	1
6	$^6He_\Lambda$ and $^6Li_\Lambda$	1	5/6	−1/2
6	$^6He_\Lambda$ and $^6Li_\Lambda$	2	1	1
7	$^7Li_\Lambda$	1/2	2/3	−1/3
7	$^6Li_\Lambda$	3/2	1	1
7	$^7He_\Lambda$ and $^7Be_\Lambda$	1/2	0	1
8	$^8Li_\Lambda$ and $^8Be_\Lambda$	1	5/6	−1/2
8	$^8Li_\Lambda$ and $^8Be_\Lambda$	2	1	1
9	$^9Be_\Lambda$	1/2	0	1
9	$^9Li_\Lambda$ and $^9B_\Lambda$	3/2	9/10	−3/5
9	$^9Li_\Lambda$ and $^9B_\Lambda$	5/2	1	1
10	$^{10}Be_\Lambda$ and $^{10}B_\Lambda$	1	5/6	−1/2
10	$^{10}Be_\Lambda$ and $^{10}B_\Lambda$	2	1	1
11	$^{11}B_\Lambda$	5/2	20/21	−5/7
11	$^{11}B_\Lambda$	7/2	1	1
12	$^{12}B_\Lambda$	1	5/6	−1/2
12	$^{12}B_\Lambda$	2	1	1
13	$^{13}C_\Lambda$	1/2	0	1

cleons and the anomalous magnetic moment of the Λ-particle with suitable numerical coefficients. The result is,

$$\mu = [I]^{\frac{1}{2}} C^{I_k I}_{I 0 I}\{(-)^{-\alpha+\eta+k-I}([\alpha]^{\frac{1}{2}}/C^{\lambda k \lambda}_{\alpha 0 \lambda}) W(\alpha I \alpha I; \eta k)\mu(A-1) + \\ + (-)^{\alpha-\eta+k-I}([\eta]^{\frac{1}{2}}/C^{\eta k \eta}_{\eta 0 \eta}) W(\eta I \eta I; \alpha k)\mu_\Lambda\}, \tag{32}$$

where $k = 0\,1\,0$, $\mu(A-1)$ and μ_Λ are the magnetic moments of $A-1$ nucleons and the Λ-particle respectively.

The magnetic moment in j-j scheme is also given simply by,

$$\mu = [J]^{\frac{1}{2}} C^{J_\alpha 1 J}_{J 0 J}\{(-)^{j'+1-J_\alpha-J}([J_\alpha]^{\frac{1}{2}}/C^{J_\alpha 1 J_\alpha}_{J_\alpha 0 J_\alpha}) W(J_\alpha J J_\alpha J; j'1)\mu(A-1) + \\ + (-)^{J_\alpha+1-j'-J}([j']^{\frac{1}{2}}/C^{j'1j'}_{j'0j'}) W(j' J j' J; J_\alpha 1)\mu_\Lambda\}. \tag{33}$$

We tabulate the coefficients appearing in (32) and (33) for each hyperfragment in Table VII.

There are no available data on the magnetic moments of the hyperfragments at this moment but it will be convenient to set up the Table mentioned above.

We can obtain some qualitative knowledge from the Table; for instance assuming positive or negative sign for the anomalous magnetic moment of the Λ-particle and combining it with Table II and the observed magnetic moment of the ordinary nucleus with mass number $A-1$ we can compute the magnetic moments and select for measurement the hyperfragment with the largest anomalous moment.

9. – Phenomenological $\mathcal{N}$-Λ potential.

We considered in Section 4 the central interaction of $\mathcal{N}$-Λ forces assuming the same radial dependence for the non-spin dependent and spin dependent part of the interaction. If we introduce the tensor and the two-body spin-orbit interaction with the same radial dependence $V(r)$ into the potential (25)

$$b(3\sigma_{\mathcal{N}z}\sigma_{\Lambda z} - \boldsymbol{\sigma}_{\mathcal{N}}\cdot\boldsymbol{\sigma}_\Lambda)V(r) \tag{34}$$

and

$$c(\boldsymbol{l}_{\mathcal{N}} + \boldsymbol{l}_\Lambda)\cdot(\boldsymbol{s}_{\mathcal{N}} + \boldsymbol{s}_\Lambda)V(r), \tag{35}$$

we can obtain the relation between b and c from the obtained parameters P^s_1 and P^t_2. Remembering that the wave function in j-j scheme (two-body) is connected with that in L-S scheme by the $\bar{X}$ coefficient [28] and the trans-

[28] J. M. Kennedy and M. J. Cliff: CRT-609 (Chalk River, Ont., 1955).

formed wave function gives the matrix element of the tensor operator in the form [29]

$$
(36)\qquad \begin{cases} (l = J \quad |S_{\mathcal{N}\Lambda}| l = J) = 2\,, \\ (l = J-1\,|S_{\mathcal{N}\Lambda}| l = J-1) = -2(J-1)/[J]\,, \\ (l = J+1\,|S_{\mathcal{N}\Lambda}| l = J+1) = -2(J+1)/[J]\,, \\ (l = J \pm 1\,|S_{\mathcal{N}\Lambda}| l = J \mp 1) = 6\sqrt{J(J+1)}/[J]\,. \end{cases}
$$

The matrix element of (35) in j-j scheme is easily written down for $j = l \pm \frac{1}{2}$ and $j' = l' \pm \frac{1}{2}$. Using these we have the following relation between the parameters P_1', P_2', and a, b, c:

$$
(37)\qquad \frac{1 - \frac{5}{3}a + \frac{2}{3}b - \frac{1}{3}c}{1 + a - \frac{2}{5}b + c} = \frac{P_1'}{P_2'}\,.
$$

Using $a = -.22$ we obtain

$$
(38)\qquad b - 1.7\,c = \quad .37 \quad \text{for the antiparallel spins}\,,
$$

$$
(39)\qquad b - .54\,c = -2.0 \quad \text{for the parallel spins}\,.
$$

10. – Discussion.

In the previous sections we have developed a theory of hyperfragments patterned after the shell model for complex nuclei; several simplifying assumptions had to be made in the course of this development like the restriction to two-particle interactions and the neglect of possible configuration mixing. But these are all assumptions of convenience and can be avoided in a more extensive calculation; they will be discussed in another paper. It was felt worth-while to spell out the predictions of the simplest model and to correlate the experimental results on this basis.

We have briefly commented on the advantages of using this model in the introductory section. The points emphasized there were that in any attempt at calculation of the properties of the hypernuclei from « first principles » one was forced to make simplifying assumptions along the way so that one eventually ends up with a model only; and that in our method the s and p shell hyperfragments were treated systematically rather than using special models for the various hypernuclides. From the comparison in the preceeding section between the predictions and the experimental binding energies one finds the

[29] J. J. DE SWART: private communication.

model to be satisfactory; if the model has some physical foundation, it should also be capable of correlating the decay data. Unlike the binding energies the decay data are much less systematic; they involve several additional parameters like the ratio of s and p amplitudes in the decay of the free Λ-hyperon and the matrix element for the non-mesic transition $\Lambda + \mathcal{N} \to \mathcal{N} + \mathcal{N}$ ([30]). For the purely charged mesic channels only the first parameter need be introduced if one makes the simplifying aussmption of neglecting multiple scattering of the decay pion. These matters are to be treated in a separate paper.

The question may arise at this point if the model is «really» true; after all, one is here forced to choose the shell model for s and p shell hypernuclei where one normally does not «expect» it to be valid. Actually, no critical evaluation of the success of the model has been made in this region of mass numbers. It is however known that the projection theorem can correlate the ground state spins, binidng energies and first excited states in the $p_{\frac{3}{2}}$ shell though for the higher excited states the agreement is poor. An exploratory study of the $p_{\frac{1}{2}}$ and $s_{\frac{1}{2}}$ shells (for ordinary nuclei) made in this laboratory shows satisfactory agreement ([31]); and it is in fact possible to infer that the poor fit for excited states came in from the increased spreading of the one-particle wave functions consequent on the loose binding of these states. These comments may show why we feel justified in using the projection theorem and two-particle interactions for the study of s and p shell hyperfragments.

* * *

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Note added in proof:

(i) According to a private communication from Prof. M. M. Block 14 $^4\mathrm{He}_\Lambda$ from the reaction (23) were observed. Correcting for efficiency of detection, this corresponds to 21 $^4\mathrm{He}_\Lambda$ which is about $21/2\,500 \simeq 1\%$ of the total number of interactions.

(ii) Both $^{11}\mathrm{Be}_\Lambda$ and $^{11}\mathrm{C}_\Lambda$ are also stable hyperfragments which have spins $\frac{1}{2}$ and the cofficients of P_1' and P_2' are 9/4 and 15/4 respectively.

([30]) This process might be important for the non-mesic decay compared to the conversion mechanism of Cheston and Primakoff: *Phys. Rev.*, **92**, 1537 (1953).

([31]) F. B. Wang: *Analysis of the binding energies of p shell nuclei by the projection theorem*. The author thanks Miss F. B. Wang for informing him of her analysis in advance of publication.

RIASSUNTO (*)

Si sviluppa un modello per la struttura degli iperframmenti, simile al modello a strati per i nuclei ordinari. Si formulano le relazioni del teorema di proiezione per iperframmenti degli strati s e p ed il caso delle interazioni di due particelle è trattato in dettaglio. I parametri di interazione per iperframmenti degli strati s e p vengono dedotti da un adattamento ai dati sperimentali; la bontà dell'approssimazione ottenuta col modello è soddisfacente. Una deduzione interessante è che $^4He_\Lambda$ ha spin zero e che quindi il kaone è pseudoscalare. Si presenta una lista degli iperframmenti comprendente i livelli di energia, la stabilità e gli spin possibili. Come ulteriore applicazione del teorema di proiezione si deriva una espressione per il momento magnetico di vari ipernuclei. Il presente modello viene confrontato con lavori precedenti e si fanno alcune considerazioni sulla sua attendibilità.

(*) *Traduzione a cura della Redazione.*

Sul decadimento beta dell'^{89}Sr.

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(ricevuto il 26 Aprile 1960)

Riassunto. — Si descrive un'esperienza diretta a individuare la forma specifica dell'interazione che regola il decadimento β dello ^{89}Sr, il cui spettro è « unico primo proibito ». I risultati, ottenuti mediante misure di correlazione angolare « elettrone-neutrino », indicano che l'interazione predominante è del tipo « vettore assiale », non escludendo tuttavia un possibile contributo della interazione di tipo « tensore » in misura non superiore al 26%.

1. – Introduzione.

Esperienze dirette a individuare la forma specifica delle interazioni che regolano il decadimento β, attraverso la misura di correlazioni angolari nel decadimento di alcuni nuclidi con transizioni « permesse », sono state effettuate recentemente [1]: i risultati indicano che l'interazione predominante è del tipo « vettore » per i decadimenti secondo Fermi, e del tipo « vettore assiale » per i decadimenti secondo Gamow-Teller.

Al fine di estendere l'indagine a transizioni « proibite », si è intrapreso uno studio sistematico dei decadimenti di nuclidi che presentano transizioni di questo tipo. Si espongono nel seguito i primi risultati relativi allo ^{89}Sr, il cui spettro è « unico primo proibito » [2].

Il decadimento dello ^{89}Sr segue lo schema:

$$^{89}_{38}\mathrm{Sr} \rightarrow {}^{89}_{39}\mathrm{Y} + \beta^- + \bar{\nu} \; ; \qquad \log ft = 8.5 \; ; \qquad E_{\max} = 1.46 \text{ MeV} \; ;$$

con variazione di spin $\Delta I = 2$, e cambio della parità nucleare.

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(1) J. S. Allen: *Rev. Mod. Phys.*, **31**, 791 (1959).

(2) L. M. Langer e H. C. Price jr.: *Phys. Rev.*, **76**, 641 (1949).

(3) E. Konopinski, in K. Siegbahn: *Beta- and Gamma-Ray Spectroscopy* (Amsterdam, 1955), pp. 292-313; J. S. Allen: *The Neutrino* (Princeton, 1958), chap. IV.

Nella teoria del decadimento β [3] le transizioni « uniche proibite » con $\Delta I = n+1$, essendo n l'ordine della transizione, risultano dipendenti da un solo elemento di matrice nucleare. In particolare, per transizioni « uniche prime proibite », il corrispondente elemento di matrice viene expresso mediante il tensore simmetrico di traccia nulla B_{ij}: le regole di selezione relative prevedono infatti:

$$\Delta I = 0\,, \quad \pm 1\,, \quad \pm 2 \ \ (\text{non } 0 \to 0,\ \tfrac{1}{2} \to \tfrac{1}{2},\ 0 \leftrightarrow 1)\,, \quad \Delta \Pi = -1$$

(interazione « vettore assiale » o « tensore »).

Per transizioni « uniche prime proibite » la probabilità per unità di tempo di osservare un elettrone nell'intervallo di energia compreso fra E ed $E+\mathrm{d}E$, ad un angolo fra ϑ e $\vartheta+\mathrm{d}\vartheta$ rispetto al neutrino, è espressa dalla relazione:

$$P(E, \vartheta)\,\mathrm{d}E\,\mathrm{d}\vartheta = C_1^{(2)}(\tfrac{1}{2}\sin\vartheta\,\mathrm{d}\vartheta)\,[(G^2/2\pi^3)\,F_0 pEq^2\,\mathrm{d}E]\,, \tag{1}$$

dove $p = (E^2-1)^{\frac{1}{2}}$ è la quantità di moto dell'elettrone, q la quantità di moto del neutrino ed F_0 la funzione di Fermi [4].

Il fattore di correzione $C_1^{(2)}$ è riferito in diverse formulazioni [5]: ammettendo inessenziali i termini con Re $(C_T C_A^* + C_T'^* C_A')$, e considerando trascurabili i termini con Im $(C_T C_A^* + C_T'^* C_A')$, sia perchè sono proporzionali ad αZ, sia perchè essi richiedono che nella transizione interferiscano interazioni di tipo VA ed STP, le funzioni di correlazione angolare per le transizioni « uniche prime proibite » assumono la forma:

$$[5 \mp (p/E)\cos\vartheta]\,D \pm pq(p/E)\sin^2\vartheta\,, \tag{2}$$

dove $D = p^2+q^2+2pq\cos\vartheta$, e i segni superiore e inferiore si riferiscono rispettivamente alle interazioni di tipo A e T. Il fattore di correzione $C_1^{(2)}$, nell'approssimazione ammessa, risulta indipendente dal campo coulombiano del nucleo: questa circostanza risulta dall'accoppiamento degli spin dei leptoni, caratteristico delle interazioni secondo Gamow-Teller.

2. – Principio della misura.

Il dispositivo sperimentale consiste di un rivelatore di elettroni, di una sorgente di ^{89}Sr e di un rivelatore di ioni, allineati sullo stesso asse (Fig .1): il sistema è contenuto in un cilindro metallico in cui viene mantenuta una

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(4) E. Greuling e M. L. Meeks: *Phys. Rev.*, **82**, 531 (1951).

(5) G. E. Lee-Whiting: *Can. Journ. Phys.*, **36**, 1199 (1958); A. M. Bincer: *Phys. Rev.*, **112**, 244 (1958).

pressione inferiore a 10^{-6} mm Hg. L'ampiezza degli impulsi prodotti dal rivelatore di elettroni e i tempi di volo degli ioni di rinculo vengono ricavati dalla

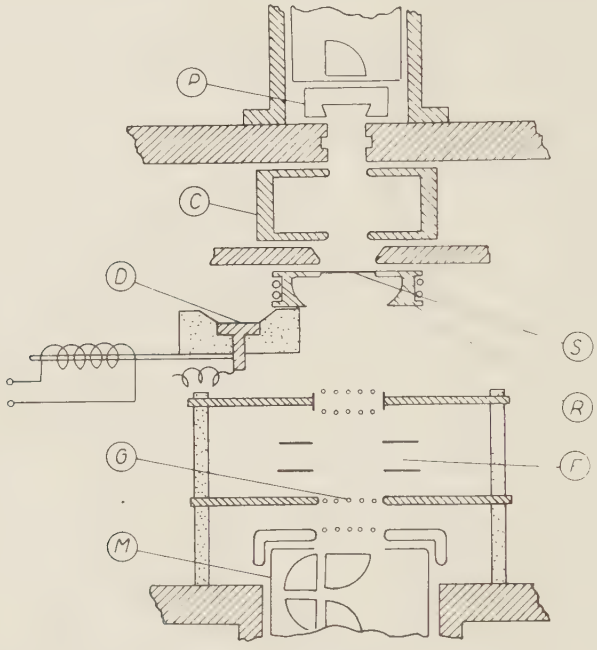

Fig. 1. – Dispositivo sperimentale: (P) scintillatore plastico; (C) collimatore degli elettroni; (S) sorgente; (D) dispositivo per la polverizzazione catodica; (R) riscaldatore; (F) e (G) collimatori e griglie acceleratrici degli ioni; (M) moltiplicatore di elettroni.

fotografia della traccia di un tubo oscillografico: è possibile pertanto determinare per ogni evento l'energia dell'elettrone e la velocità del corrispondente ione di rinculo.

Il fattore di correlazione angolare (2) favorisce l'emissione della coppia elettrone-neutrino nello stesso emisfero: in particolare, le interazioni di tipo T presentano un'asimmetria più marcata rispetto alle corrispondenti

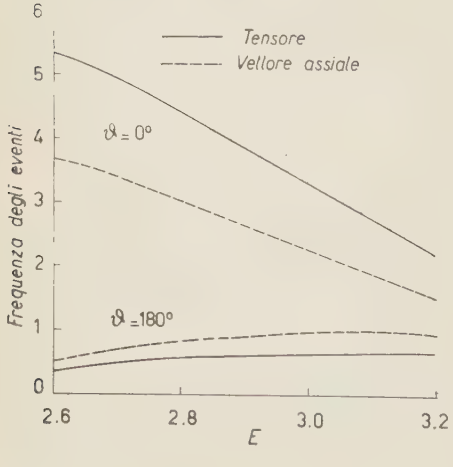

Fig. 2. – Frequenza degli eventi di coincidenza elettrone-ione di rinculo in funzione dell'energia totale dell'elettrone, in unità mc^2, per $\vartheta_{ev}=0°$ e $\vartheta_{ev}=180°$, prevista per interazioni di tipo « tensore » e « vettore assiale ».

di tipo A. In Fig. 2 sono riportate le curve rappresentative degli eventi con $\vartheta = 0°$ e $\vartheta = 180°$ per le interazioni di tipo T ed A: risulta peraltro che la asimmetria è meno accentuata alle energie più alte. Il confronto fra eventi concordi ($\vartheta = 0°$) e discordi ($\vartheta = 180°$) fornisce un metodo di discriminazione per i due tipi di interazione.

3. – Dispositivo sperimentale.

Il dispositivo è illustrato in Fig. 1. La sorgente (S) è costituita da un sottile strato di ^{89}Sr su pellicola di teflon ($(0.5 \div 0.7)$ mg/cm^2) previamente metallizzata: il deposito viene effettuato mediante polverizzazione catodica di $SrCl_2$ depositato su una lamina di platino. Quest'ultima è fissata ad un sostegno di teflon che può essere sospinto con un elettromagnete in posizione opportuna per la polverizzazione (D).

La polverizzazione catodica permette di ottenere una distribuzione omogenea ed una regolazione accurata dell'intensità della sorgente.

Per evitare l'adsorbimento di gas, la sorgente è mantenuta a temperatura conveniente mediante un anello coassiale di alluminio (R) riscaldato elettricamente. Questo è sagomato in modo da consentire un riscaldamento omogeneo della pellicola.

L'esperienza è stata eseguita con sorgenti depositate rispettivamente su Pt, Ag e W: i risultati mostrano peraltro che le superfici di W ed Ag sono più convenienti ai fini della misura di quelle di Pt. La sorgente veniva rinnovata ogni 30 ore per prevenire possibili alterazioni della superficie attiva, ed anche per eliminare eventuali differenze dipendenti dalla struttura delle singole sorgenti: risulta infatti che sorgenti preparate con procedimento apparentemente identico non presentano sempre identiche caratteristiche.

Il rivelatore di ioni (M) consiste di un moltiplicatore di elettroni a 10 dinodi: un diaframma collima il fascio ionico entro un angolo di $\pm 3°$. La taratura del dispositivo per la misura dei tempi di volo degli ioni è stata effettuata per mezzo di un

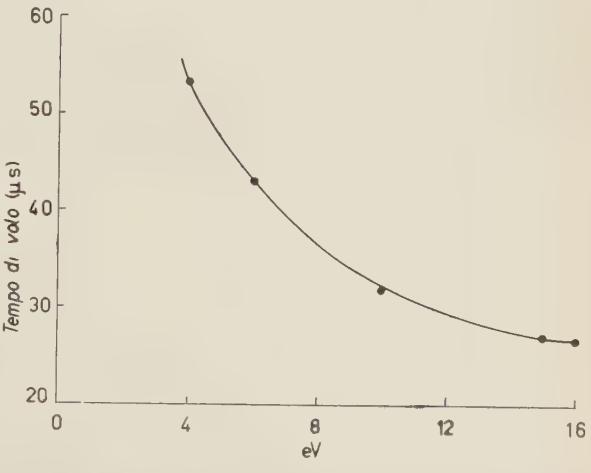

Fig. 3. – Valori misurati e spettro previsto dei tempi di volo degli ioni W^+ in funzione della loro energia.

analizzatore a deflessione elettrostatica pulsato, con sorgente ionica di tipo Penning, per gli ioni leggeri; e per mezzo di una sorgente ionica a filo caldo pulsata, per gli ioni pesanti. I risultati dimostrano che i tempi di volo concordano, per le diverse energie, con i valori previsti (Fig. 3).

Il rivelatore di elettroni consiste di un collimatore (C), di uno scintillatore plastico (P) e di un fotomoltiplicatore di tipo 50 AVP selezionato.

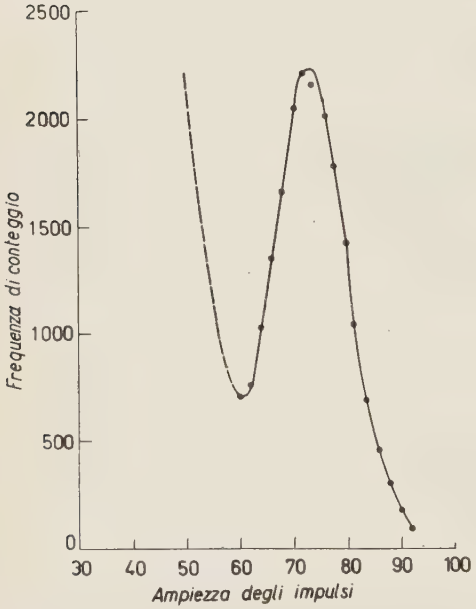

Fig. 4. – Spettro degli elettroni di conversione del ^{137}Cs.

La forma del collimatore è tale da attenuare gli effetti di bremsstrahlung e diffusione degli elettroni; l'apertura del diaframma definisce il fascio elettronico entro un angolo di $\pm 3°$.

Lo scintillatore plastico è sagomato in modo da limitare gli effetti di back-scattering degli elettroni incidenti; la sua superficie è resa otticamente diffondente.

La taratura dello spettrometro è stata effettuata mediante gli elettroni di conversione interna del ^{137}Cs (Fig. 4) e mediante i punti terminali dei diagrammi di Kurie degli spettri β di alcuni nuclidi; i risultati indicano, per l'intervallo considerato, che l'andamento dell'ampiezza degli impulsi è approssimativamente lineare con l'energia degli elettroni. La larghezza a mezzo massimo dello spettro degli elettroni di conversione interna del ^{137}Cs risulta di 120 keV; ammettendo che la distribuzione degli impulsi risultanti dallo scintillatore, per elettroni monoenergetici, si approssimi ad una funzione gaussiana, si trova che la risoluzione dello spettrometro vale $0.15\, W^{-\frac{1}{2}}$, dove W è l'energia cinetica degli elettroni espressa in MeV ([6]).

4. – Risultati.

Le quantità di moto degli ioni di rinculo devono soddisfare, nello schema sperimentale adottato, alla relazione:

$$r = (E^2 - 1)^{\frac{1}{2}} \pm (E_{max} - E)\,, \tag{3}$$

([6]) W. J. PRICE: *Nuclear Rad. Detection* (New York, 1958), p. 199.

dove il segno superiore vale per un angolo $\vartheta = 0°$ fra le direzioni di emissione della coppia elettrone-neutrino e l'inferiore per $\vartheta = 180°$ (Fig. 5).

La quantità di moto degli ioni di rinculo può essere alterata da effetti inerenti alla superficie sulla quale è deposto lo ^{89}Sr; se tuttavia si ammette che tali effetti variano poco con l'energia dello ione, entro un intervallo limitato, è possibile fare una selezione degli eventi, stabilendo di considerare validi quegli eventi che risultano in accordo con la relazione (3), entro i limiti di precisione delle misure, quando sia sottratto il fondo relativo alle coincidenze casuali e spurie. È quindi possibile distinguere fra gli eventi validi quelli relativi ad emissione concorde e discorde, in base alla conoscenza dell'energia dell'elettrone e della quantità di moto del corrispondente ione di rinculo (Fig. 5).

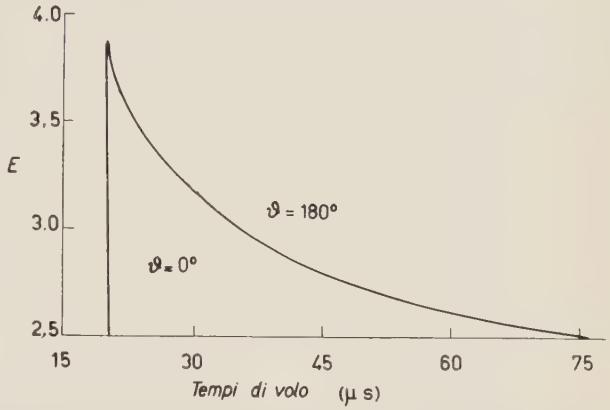

Fig. 5. – Energia degli elettroni di decadimento dello ^{89}Sr in funzione dei tempi di volo degli ioni di rinculo ^{89}Y, per $\vartheta_{ev}=0°$ e $\vartheta_{ev}=180°$.

Il numero delle coincidenze di fondo cresce rapidamente al decrescere dell'energia degli elettroni. In prossimità del limite superiore dello spettro si annulla invece la differenza fra i tempi di volo degli ioni corrispondenti ad eventi concordi e discordi. Si è perciò ritenuto opportuno limitare la considerazione agli eventi compresi fra $E_1 = 2.6$ ed $E_2 = 3.2$.

Tabella I.

ΔE	2.6÷2.8	2.8÷3.0	3.0÷3.2
T_c/T_d	10.76	6.47	4.25
A_c/A_d	5.04	2.98	1.95
Valori sperim.	5.6±2.5	3.1±1.4	2.1±0.8

In Tabella I sono riportati i valori calcolati dei rapporti fra eventi concordi di tipo T_c e A_c e i corrispondenti discordi T_d e A_d per tre intervalli energetici compresi fra E_1 ed E_2, in confronto coi valori sperimentali. Da essi si deduce

che, per il decadimento β dello ^{89}Sr, l'interazione predominante è del tipo « vettore assiale ». I risultati non permettono tuttavia di escludere un possibile contributo dell'interazione di tipo « tensore » in misura non superiore al 26%.

* * *

Ringraziamo il Prof. A. Rostagni ed il Prof. S. Fubini per il loro interessamento e per le utili discussioni sull'argomento del presente lavoro.

SUMMARY (*)

An experiment for the determination of the specific shape of the interaction which controls the β-decay of ^{89}Sr, whose spectrum is « unique first forbidden », is described. The results, obtained using angular correlation « electron-neutrino » measurements, show that the prevailing interaction is of the « axial vector » kind, without excluding, however, a possible contribution from the interaction of « tensor » kind not greater than 26%.

(*) *Editor's Translation.*

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The Gamma Ray Spectra of ^{65}Ni and ^{65}Zn; Characteristics of the Lower Excited States of ^{65}Cu.

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Summary. — The γ-ray spectra of ^{65}Ni and of ^{65}Zn were studied by means of scintillation techniques. It is shown that the first excited state of ^{65}Cu at 770 keV is not populated in either of the decays. In ^{65}Ni (half-life: (2.50 ± 0.03) h) two new γ-rays of (1.63 ± 0.05) and (1.73 ± 0.05) MeV were found, which originate at levels of the same energy known to exist in ^{65}Cu. Evidence is presented that the level at 1.482 MeV does not belong to the quartet of levels arising from the coupling of the 29th proton ($p_{\frac{3}{2}}$) to the ^{64}Ni-core in its first excited 2^+ state (at 1.34 MeV), as assumed by Lawson and Uretsky in their application of the center-of-gravity theorem to ^{65}Cu. Agreement with this theorem is obtained for certain spin assignments to the other four excited states in ^{65}Cu below 2 MeV.

1. – Introduction.

The level structure of ^{65}Cu has been studied in great detail at excitation energies below 3.1 MeV by means of nuclear reactions. Foremost among these studies is the one performed by Mazari, Buechner and De Figueiredo [1],

(1) M. Mazari, W. W. Buechner and R. P. De Figueiredo: *Phys. Rev.*, **108**, 373 (1957).

who report (at least) 21 excited states below 3.08 MeV from scattering of protons by ^{65}Cu. In the region below 2 MeV levels are found at 770, 1114, 1482, 1623 (all ± 4 keV) and (1725 $\pm$ 6) keV. The results of experiments by other groups ([2]) are generally in agreement with those mentioned above, with the exception of the energy of the first excited state as reported by TEMMER and HEYDENBURG ([3]), who find (815 $\pm$ 8) keV from Coulomb excitation studies with α-particles.

Some of the lower lying levels are also populated in the β-decay of the two neighbouring nuclei. The positon and capture transitions from ^{65}Zn lead to the ground state and to the level at 1114 keV, while the negaton decay of ^{65}Ni has been reported to proceed to the ground state and to the levels at 1114 and 1482 keV.

It is at first sight somewhat surprising that the first excited state in ^{65}Cu does not seem to be populated in either of the two decay modes. This fact might be explained by invoking the selection rules on β and γ transitions, which in turn would aid in assigning a spin to the 770 (or 815) keV level.

The nuclide ^{65}Cu belongs to the class of nuclei to which the center-of-gravity theorem ([4]) has been applied. A knowledge of the spins of the levels involved in this theorem would put the application to ^{65}Cu on a somewhat firmer footing.

For this reason a search was made for the possible occurrence of a 770 (or 815) keV γ-ray in the decay of both ^{65}Ni and ^{65}Zn and for the possible existence of additional weak γ-rays in the decay of ^{65}Ni.

2. – Production, chemical procedures and measurements.

The radioactive sources were produced in the synchrocyclotron of the Instituut voor Kernphysisch Onderzoek in Amsterdam. The ^{65}Zn was formed by 21 MeV deuteron bombardment of copper (30 μAh) and the zinc fraction extracted with chloroform as zinc di-β-naphtylthiocarbazone compound ([5]). The ^{65}Ni sources were produced by the (d, 2p) reaction on copper with 25 MeV deuterons (≈ 5 μAh). The irradiated copper target was dissolved in nitric acid, nickel and zinc carriers were added and the nickel fraction was precipitated with dimethylglyoxim in an alcoholic solution to which ammonia was added. Repeated application of solution of the precipitate in hydrochloric acid, fol-

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([2]) Nuclear Data Sheets, NRC 58-1-34, ^{65}Cu-1; National Research Council, Washington, D.C.

([3]) G. M. TEMMER and N. P. HEYDENBURG: *Phys. Rev.*, **104**, 967 (1956).

([4]) R. D. LAWSON and J. L. URETSKY: *Phys. Rev.*, **108**, 1300 (1957).

([5]) L. LINDNER and G. A. BRINKMAN: *Physica*, **21**, 747 (1955).

lowed by precipitation as nickel dimethylglyoxim was made, until the filtrate remained inactive. It was found necessary to use electrolytic copper as the target, since the bismuth, produced from the lead impurity in normal copper, tended to be present in the nickel precipitates.

The γ-ray spectra of the various nickel sources produced were measured at the Instituut voor Kernphysisch Onderzoek (in view of their short half-life of 2.5 hours) with a 62 mm high by 62 mm diameter cylindrical NaI(Tl) Harshaw crystal, provided with a well (31 mm deep, 9 mm diameter), optically coupled to a DuMont 6363 photomultiplier tube. This combination had a resolution of 7.6% for the ^{137}Cs line; it was calibrated for efficiency. The spectra were recorded on a 100 channel RIDL pulse height analyzer. The sources were measured at various distances from the crystal in order to recognize possible residual summing effects. Some information on coincidences was also obtained by placing the sources inside the well of the crystal.

All results of this series of measurements were analyzed at the Nuclear Spectroscopy Laboratory in Naples (*).

The long-lived ^{65}Zn (half-life 245 days) sources were measured at the Nuclear Spectroscopy Laboratory in Naples. The experimental set-up consisted in this case of a 76 mm high by 76 mm diameter NaI(Tl) Harshaw crystal of the well type, coupled to a 6363 DuMont photomultiplier tube (resolution 7.8% for the 662 keV line); this combination was also calibrated for efficiency. The pulse height distribution was measured with a Franklin model 348 single channel analyzer of the Fairstein type.

3. – Experimental results.

The pulse height distributions from ^{65}Ni are shown in Fig. 1, Table I lists the energies and the relative intensities of the various γ-rays of ^{65}Ni resolved from the pulse height distribution; the results obtained by other groups are also reported. The half-life of ^{65}Ni was determined to be (2.50 ± 0.03) hours from the decay of the two prominent peaks (over a period of 9 half-lives). The γ-rays of (1630 ± 50) and (1730 ± 50) keV obviously originate at the levels of (1623 ± 4) and (1725 ± 6) keV, known to exist in ^{65}Cu.

Additional γ-rays can be expected to exist in the decay of ^{65}Ni, corresponding to transitions between the known levels of ^{65}Cu. A special search was made for these γ-rays by accurately analyzing the total pulse height distribution. Calibration peaks of ^{139}Ce (165 keV), ^{203}Hg (279 keV), ^{113}Sn (393 keV),

(*) Istituto Nazionale di Fisica Nucleare, Sottosezione di Napoli and Istituto di Fisica Superiore dell'Università.

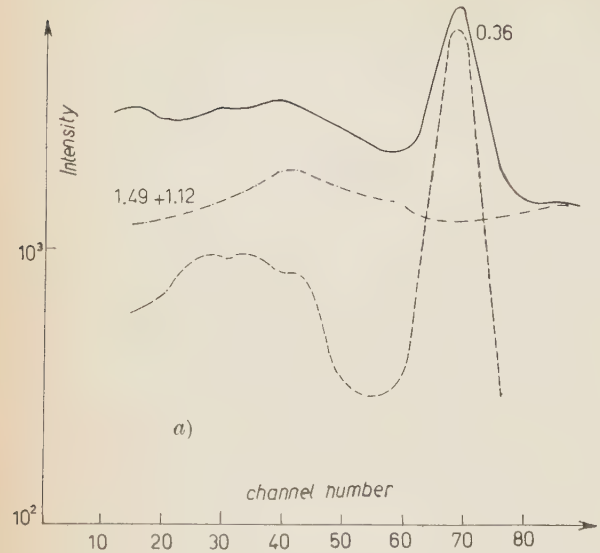
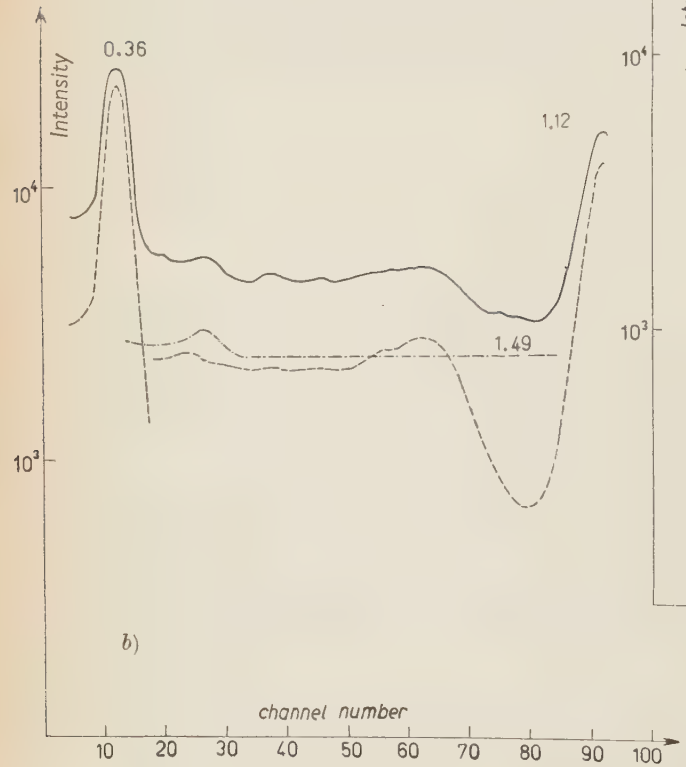


Fig. 1. – γ-ray spectrum of ^{65}Ni in three different energy regions: *a*) region below 500 keV; *b*) region between 300 and 1 200 keV; *c*) region above 700 keV. The spectra were recorded with a source-detector distance of 15 cm. The pulse heigth distributions corresponding to the individual γ-rays are shown as dashed curves.



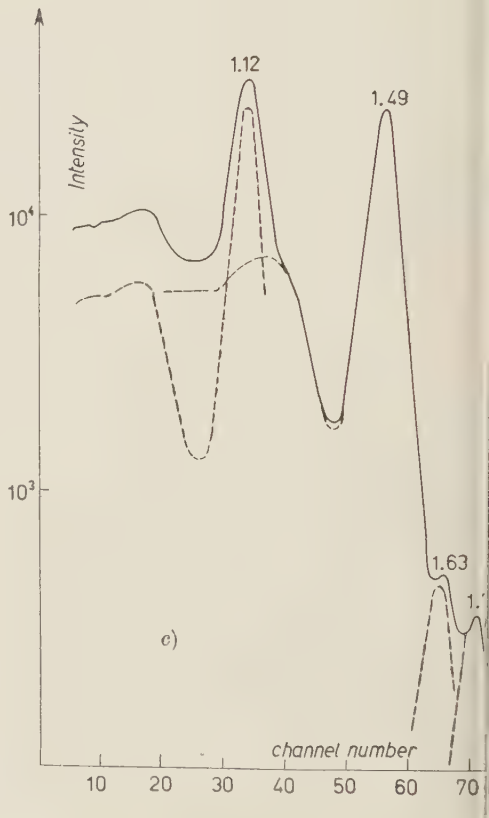


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TABLE I. – *Energy values and relative intensities of the γ-rays from* 2.50 h ^{65}Ni.

Ref. (6); sl, pe		Ref. (7); scin		Ref. (8); scin		Ref. (9); scin		This work; scin	
Energy (keV)	Rel. int.	Energy (keV)	Rel. int.	Energy (keV)	Rel. int.	Energy (keV)	Rel. int.	Energy (keV)	Rel. int.
370	34	360±20	37	378± 6	20	370± 5	16	360±10	33±3
1120	100	1090±50	100	1110±10	100	1100±20	100	1120±40	100
1490	200	1450±70	139	1490±20	100	1460±20	147	1490±50	156±8
								1630±50	3.7±0.3
								1730±50	3.4±0.3

(6) K. SIEGBAHN and A. GOSH: *Phys. Rev.*, **76**, 307 (1949); *Ark. Mat. Astr. Fys.*, A **36** no. 19 (1949).
(7) R. L. HEATH: IDO-16291 (1956) 41; quoted on Nuclear Data Sheet NRC 59-2-24; ^{65}Ni-1.
(8) T. WIEDLING and A. CARLSSON: *Phys. Rev.*, **83**, 181 (1951); T. WIEDLING: *Thesis* (University of Stockholm, 1956).
(9) G. N. SALAITA, W. G. CANTRELL, D. L. EDENS, O. T. GOINES and A. MUKERJI: *Bull. Am. Phys. Soc.*, **4**, no. 2 (1959), 99, LI.

^{85}Sr (513 keV), ^{22}Na (511 and 1276 keV), ^{207}Bi (570 and 1064 keV), ^{137}Cs (662 keV), ^{54}Mn (834 keV) and ^{88}Y (900 and 1840 keV) were measured in the same geometries as those used for the ^{65}Ni sources. From the shapes of these standard peaks those corresponding to the γ rays present in the decay of ^{65}Ni were constructed by interpolation. The pulse height distribution of ^{65}Ni was then analysed into its component parts by successively peeling off the pulse height distributions of the various γ rays, starting at the high energy end.

In this way at first a weak peak was resolved at (500 ± 30) keV, which is of interest, since it would fit between the 1623 and 1114 keV levels. However, a more detailed analysis by comparison with the pulse height distributions due to the 1.52 MeV γ-ray from ^{42}K and the 1.84 MeV γ-ray from ^{88}Y, showed that (within the error limits) it was entirely due to the pair peak of the 1.49 MeV photo peak. No further γ rays have been found and upper limits for the relative intensities of most of the expected transitions are collected in Table II.

A decay scheme consistent with the energies of the known levels in ^{65}Cu and with the observed γ-rays from 2.5^{h} ^{65}Ni is shown in Fig. 3*a*. As a further check on this decay scheme the structure of the high energy part of the spectrum was studied with the source in different geometries. No change in structure or relative intensity could be detected (except, of course, in the 1.49 MeV peak). Since weak undetected cascades from the two upper levels would show up through an increase in the intensity of the weak peaks at 1.63 and 1.73 MeV, we can put an upper limit of the intensities of all stop-

TABLE II. – *Upper limits on the relative intensities for some unobserved γ-rays from* ^{65}Ni.

Serial numbers of levels involved	5-3	5-2	5-1	4-2	4-1	3-1	1-0
Energy (keV)	243	611	955	509	853	712	770
Relative intensity (*)	1.7	0.5	0.5	1.0	1.0	1.0	1.0

No cascades from 1623 and 1725 keV levels (<1.0).
(*) Relative to 100 (equivalent to 12 %) for the 1114 keV γ-ray.

over γ-rays from these two highest levels of $\approx \frac{1}{4}$ of that of the two high energy transitions.

The γ-ray spectrum of the ^{65}Zn sources was analysed in the same manner as described above. Standard calibration pulse height distributions due to ^{51}Cr (325 keV), ^{113}Sn (393 keV), ^{85}Sr (513 keV), ^{22}Na (511 and 1276 keV), ^{137}Cs (662 keV), ^{54}Mn (834 keV) and ^{207}Bi (570 and 1064 keV) were employed. Special attention was paid to the region where a 770 (or 815) keV photo peak might be expected to occur (see Fig. 2). A search was also made for a 340 (or 300) keV stop-over γ-ray. In this case the shape of the pulse height distribution of the 511 keV transition was deduced from that of the 513 keV transition in ^{85}Sr. Care was

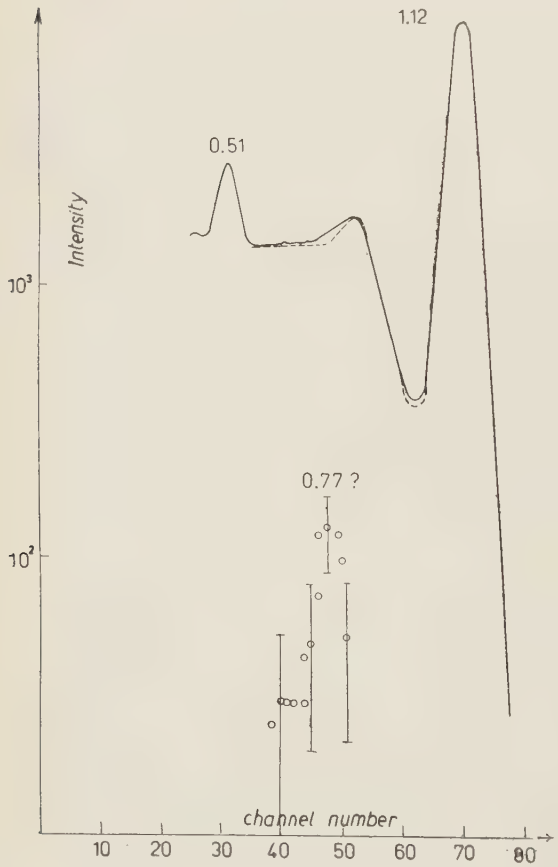

Fig. 2. – Pulse height distribution of ^{65}Zn measured with the single channel analyzer. The pulse height distribution corresponding to a single γ-ray of 1114 keV is shown as a dashed curve. In this measurement a γ-ray of 770 keV was resolved, while in other measurements the indications for its existence was less clear. Therefore the intensity value from this figure was accepted as the upper limit.

taken to annihilate all positons of ^{65}Zn by using appropriate aluminium absorbers; these same absorbers were then also used with ^{85}Sr. The relative intensities of the γ-rays observed and upper limits on the intensities of unobserved γ-rays are collected in Table III.

TABLE III. – *Relative intensities of γ-rays in the decay of* ^{65}Zn.

Energy (keV)	1114	770	340	$\gamma^{\pm}$
Relative intensity	100 (*)	$\leqslant 1$	$\leqslant 0.5$	6.6 ± 0.6

(*) Equivalent to 49%.

4. – Discussion.

The experimental knowledge concerning the levels in ^{65}Cu, the γ-ray transitions between them and their population through β-decay of the two neighbouring nuclides, is summarized in Fig. 3.

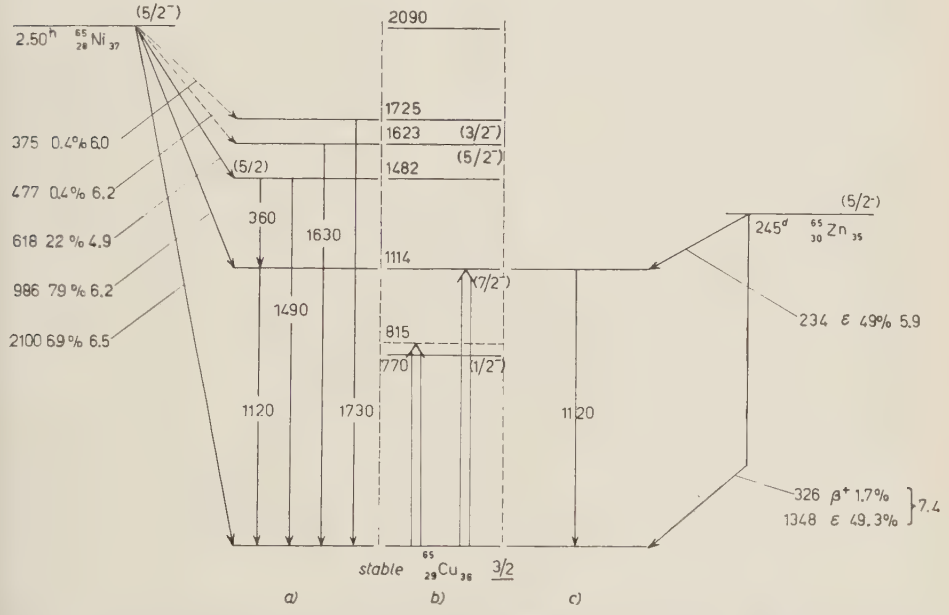

Fig. 3. – Experimental knowledge concerning ^{65}Ni, ^{65}Cu and ^{65}Zn. Fig. 3*a* shows the decay scheme of ^{65}Ni, Fig. 3*b* the level scheme of ^{65}Cu and Fig. 3*c* the decay scheme of ^{65}Zn.

From the decay scheme adopted for ^{65}Ni and by assuming that 69% of all β-decays lead directly to the ground state (7), the intensities and log *ft*-

values of the other β branches can be calculated; those values are collected in Table IV.

TABLE IV. – *Intensities and* log *ft-values for the various β branches in the decay of* ^{65}Ni *and of* ^{65}Zn.

Final level (keV)	^{65}Ni			^{65}Zn		
	E_{β^-} (keV)	Int. (%)	log *ft*	E_ε (keV)	Int. (%)	log *ft*
0	(2100±100)	(69)	6.5	1348±2	$\begin{cases}\beta^+ & 1.7\\ \varepsilon & 49.3\end{cases}$	7.4
770	1330±100	< 0.12	> 8.4	578±4	⩽0.5	⩾8.5
1114	986±100	7.9±0.4	6.2±0.1	234±4	49	5.9
1482	618±100	22±1	4.9	—	—	—
1623	477±100	0.44±0.04	6.2±0.2	—	—	—
1725	375±100	0.40±0.04	6.0±0.3	—	—	—

The most striking result of this study is the absence of the 770 keV γ-ray in the decays of both ^{65}Zn and ^{65}Ni. The lower limit on the log *ft*-value (8.5) for a possible capture branch in ^{65}Zn to the 770 keV level indicates a second forbidden transition (unless one assumes an even stricter *l*-forbiddenness than that for the ground state transition from ^{65}Zn). A spin $\frac{1}{2}^-$ is therefore indicated for the first excited state of ^{65}Cu. Essentially the same conclusion is drawn from the lower limit on the log *ft*-value (> 8.4, see Table IV) of the possible negaton branch from ^{65}Ni to the same final level.

The spins of the other four excited levels in ^{65}Cu below 2 MeV are more difficult to assign. If one accepts a ground state spin of $\frac{5}{2}^-$ for ^{65}Ni (the shell model predicts $f_{\frac{5}{2}}^{-1}$ for the configuration of ^{65}Ni with 37 neutrons) all four levels should have spins $\frac{3}{2}^-$, $\frac{5}{2}^-$ or $\frac{7}{2}^-$.

Usually a spin $\frac{7}{2}^-$ is assigned to the 1114 keV level, but there do not seem to be any strong arguments for this. The measured total conversion coefficients ([10]) for the 1114 keV γ-ray range from $(1.8 \pm 0.2)\cdot 10^{-4}$ to $(2.3 \pm 0.3)\cdot 10^{-4}$. The theoretical values (α_K and α_L taken from ROSE ([11]), $\alpha_M + \alpha_N + \ldots$ estimated as between $\frac{1}{3}$ and $\frac{1}{4}$ of α_L) for a pure *M*1, pure *E*2 and pure *M*3 transition are 1.75, 1.95 and $7.58\cdot 10^{-4}$ respectively. Therefore the experimental values are compatible with anything from pure *M*1 to a mixed transition consisting of 88 % *E*2 + 12 % *M*3. In fact, the most accurate measurement ([12]) ($\alpha_{tot} = (2.2 \pm 0.1)\cdot 10^{-4}$) yields an admixture of (5 ± 2) % *M*3. However,

([10]) Nuclear Data Sheets, NRC 59-2-27, ^{65}Zn-2, National Research Council, Washington, D.C.

([11]) M. E. ROSE: *Internal Conversion Coefficients* (Amsterdam, 1960).

([12]) J. F. PERKINS and S. K. HAYNES: *Phys. Rev.*, **92**, 687 (1953).

competition between $E2$ and $M3$ must be excluded on theoretical grounds: the WEISSKOPF estimate allows at most $2 \cdot 10^{-3}$% $M3$ admixture, while empirical estimates [13] reduce this by a factor 1 to 100 or even further if the transition is assumed to be mainly of a collective character. It must therefore be concluded that the experimental situation with respect to the conversion coefficient of the 1114 keV transition is still uncertain and warrants further study.

That the 1114 keV transition in ^{65}Cu contains an appreciable $E2$ part follows from the work of TEMMER and HEYDENBURG [3] on Coulomb excitation. These authors report a value of 0.027 ± 0.004 for $\varepsilon B(E2)$ for this transition. Since conversion and branching from the 1114 keV level are unimportant, $\varepsilon = 1$. Comparison of the measured $B(E2)$ value with the WEISSKOPF single particle value shows an enhancement by a factor of ≈ 10. If the 1114 MeV γ-ray is a mixed $M1+E2$ transition, with the $E2$ part being entirely of a collective character, the empirical $E2/M1$ ratio [13] is expected to lie between 30 and 0.3. Again this does not decide between a spin $\frac{7}{2}^-$ or $\frac{5}{2}^-$ for the 1114 keV level.

The angular correlation between the 370 and 1114 keV γ-rays has been measured by HARTMANN and ASPLUND [14]. These authors, assuming a pure $E2$ 1114 keV transition (spin $\frac{7}{2}^-$ for the 1114 keV level), deduce a spin $\frac{5}{2}$ for the 1482 keV level. A re-examination of these results by the present authors has shown that they are also compatible with a spin $\frac{5}{2}^-$ for the 1114 keV level and with various $M1/E2$ mixing ratios in both the 370 and 1114 keV γ-ray transitions. The limit on the branching ratio from the 1114 keV level (absence of 340 and 770 keV γ-rays) does not help in fixing the spin of the 1114 keV: the only combination that is definitely excluded is a mainly $M1$ 340 keV transition, but this only eliminates a spin $\frac{3}{2}^-$ for the upper level.

The above analysis of the published data therefore leads to the conclusion that the spin of the 1114 keV level in ^{65}Cu can equally well be $\frac{5}{2}^-$ or $\frac{7}{2}^-$.

The log ft-values of the five negaton branches from ^{65}Ni fall into two groups. From this it follows that the configuration of the ground state and of the second, fourth and fifth excited state are probably very similar, while that of the third level at 1482 keV is quite different. The rather high log ft-value for the ground state β transition is probably explainable as due to l-forbiddenness since the ground state configuration of ^{65}Ni probably contains a large fraction of $f_{\frac{5}{2}}{}^1$ (subshell closure at 38 neutrons). This same l-forbiddenness would then follow for the other β transitions with log ft between 6.0 and 6.5. The level at 1482 keV seems to possess more of a single particle structure, $f_{\frac{5}{2}}^1$ being the most likely candidate. The resulting estimate on the

[13] A. H. WAPSTRA, G. J. NYGH and R. VAN LIESHOUT: *Nuclear Spectroscopy Tables* (Amsterdam, 1959), chap. 6, Sect. 3.

[14] B. HARTMANN and I. ASPLUND: *Ark. f. Fys.*, **13**, 339 (1958).

$p_{\frac{3}{2}}$-$f_{\frac{5}{2}}$ single particle splitting is not in disagreement with NUSSBAUM's [15] estimate.

LAWSON and URETSKY [16], in their discussion of the center-of-gravity theorem as applied to the levels in ^{65}Cu, arising from the coupling of a proton in a single particle $p_{\frac{3}{2}}$ state to the first excited state of the ^{64}Ni core (at (1346 ± 14) keV), assign the following spin values: 770 keV: $\frac{1}{2}^-$; 1114 keV: $\frac{5}{2}^-$, 1482 keV: $\frac{7}{2}$ and 1623 keV: $\frac{3}{2}^-$. On the basis of the results quoted above, the 1482 keV level would have to be excluded from the quartet, while that at 1725 keV would belong to it. In order to check whether this new interpretation is also compatible with the center-of-gravity theorem, various combinations of spin values were tried. It was found that there is unfortunately quite some leeway in the spin assignment if only the energy values of the five

TABLE V. – *Energy value of the first excited* 2^+ *state of* ^{64}Ni, *calculated on the basis of the center-of-gravity theorem, from the energies of the lower levels of* ^{65}Cu *for various spin combinations.* Upper part: calculated value in agreement with experimental value within the error limits; lower part: agreement within twice the error limits.

Level (keV)					Calculated $E(2^+)$
770	1114	1482	1623	1725	
1/2	7/2	—	5/2	3/2	1355
1/2	7/2	3/2	—	5/2	1337
1/2	5/2	3/2	7/2	—	1357
1/2	5/2	7/2	—	3/2	1349
3/2	1/2	5/2	7/2	—	1359
3/2	1/2	7/2	5/2	—	1345
3/2	5/2	—	1/2	7/2	1341
5/2	1/2	—	3/2	7/2	1357
5/2	1/2	—	7/2	3/2	1337
5/2	—	3/2	7/2	1/2	1349
1/2	5/2	7/2	3/2	—	1329
1/2	7/2	—	3/2	5/2	1365
3/2	1/2	7/2	—	5/2	1376
3/2	5/2	1/2	—	7/2	1326
5/2	1/2	3/2	—	7/2	1329
5/2	—	1/2	7/2	3/2	1373
5/2	—	7/2	1/2	3/2	1331
5/2	—	7/2	3/2	1/2	1321
Experimental $E(2^+)$: 1346 ± 14					Error: ±3

[15] R. H. NUSSBAUM: *Rev. Mod. Phys.*, **28**, 423 (1956).
[16] R. D. LAWSON and J. L. URETSKY: *Phys. Rev.*, **108**, 1300 (1957).

levels are taken into account (as already pointed out in ref. ([16])). In fact, of the 120 possible spin combinations, 10 yield a calculated value for the energy of the 2^+ level in ^{64}Ni in agreement with the reported experimental value ([17]) of (1346 ± 14) keV. These 10 combinations are shown in the upper part of Table V. Four of these still yield acceptable results even if the 1482 keV level is left out. If one imposes the further condition that the spin of the 770 keV level is certainly $\frac{1}{2}^-$, only one acceptable combination is left. The combination originally proposed by LAWSON and URETSKY ([16]) falls just outside the reported error limits. If the error limits are doubled 8 more combinations become acceptable. These are listed in the lower half of Table V; the first one is the one originally proposed, the second one is another combination which leaves out the 1482 keV level and assigns a spin $\frac{1}{2}^-$ to the 770 keV level.

The additional argument on the selection of levels belonging to the quartet, derived from the log *ft*-values (if applicable) may therefore provide a restriction on the number of acceptable combinations of spin assignments. The spin sequence within the quartet thus seems to be different from the one derived ([18]) for the levels in ^{63}Cu.

The γ-ray transitions from the members of the first quartet to the ground state should contain an appreciable amount of a collective $E2$ transition, while the interquartet transitions should consist mainly of single particle $E2$ and/or (inhibited) $M1$ transitions.

From the levels at 1114, 1623 and 1725 keV ground state transitions are observed. The upper limit on the intensity of the 770 keV γ-ray implies the same upper limit on all possible γ transitions feeding it; in addition some more upper limits on possible transitions have been obtained (see Tables II and III). The upper limits on the branching ratios from these three states are in general compatible with those for pure $M1$ and for pure $E2$ transitions. They are also in agreement with the empirical ratios for normal or collective $E2$ stop-over transitions in competition with single particle $M1$ cross-over γ-rays. The inverse assignment (*i.e.* $E2$ for the cross-over and $M1$ for the stop-over transition) is only acceptable if the $E2$ transition is of a collective nature and/or the $M1$ transition is inhibited.

Therefore, the branching ratios from the three levels at 1114, 1623 and 1725 keV cannot be considered to shed any light on the character of these levels.

The branching ratio from the 1482 keV level to the three states below it is best explained by assigning a spin $\frac{5}{2}^-$ to the initial level and mainly $M1$

([17]) B. S. DŽELEPOV and YU. V. HOL'NOV: *Suppl. Nuovo Cimento*, **3**, 49 (1956).
([18]) R. A. RICCI, R. K. GIRGIS and R. VAN LIESHOUT: *Nuovo Cimento*, **11**, 156 (1959).

multipolarity to the 1482 and 370 keV γ-rays, since otherwise it would be somewhat difficult to understand the absence of 710 and 770 keV transitions.

A final point should be mentioned. The high log ft-value for the ground state transition from ^{65}Zn is usually explained as being due to l-forbiddenness, since the transition is assumed to take place between a relatively pure $f_{\frac{5}{2}}$ state and a pure $p_{\frac{3}{2}}$ state. From the point of view of purity of single particle wave functions, this interpretation should apply even more to the ground state β transition from ^{65}Ni, since there the configuration is probably purely $f_{\frac{5}{2}}^{-1}$ (subshell closure at 38 neutrons). However, this last β transition is far less inhibited than the one from ^{65}Zn, showing that an explanation on the basis of l-forbiddenness is perhaps not quite so straightforward in the case of ^{65}Zn.

* * *

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RIASSUNTO

Si sono studiati gli spettri γ associati al decadimento radioattivo del ^{65}Ni e dello ^{65}Zn con tecniche di scintillazione. Si mostra che il primo livello eccitato del ^{65}Cu a 770 keV non è popolato da nessuno dei due decadimenti. Nel ^{65}Ni (periodo di dimezzamento (2.50 ± 0.03) ore) si sono trovate due nuove transizioni γ di energia (1.63 ± 0.05) e (1.73 ± 0.05) MeV che hanno origine dai livelli aventi la stessa energia esistenti nel ^{65}Cu. La struttura di livelli del ^{65}Cu viene discussa in relazione al teorema del centro di gravità in spettroscopia nucleare. Si fa presente l'evidenza che il livello a 1.482 MeV non appartiene al quartetto di livelli ottenuti per accoppiamento del 29° protone $(p_{\frac{3}{2}})$ al core del ^{64}Ni nel suo primo stato eccitato 2^+ (a 1.34 MeV), come invece supposto da Lawson e Uretsky nell'applicazione del loro teorema al caso del ^{65}Cu. L'accordo con il teorema si ottiene per certe assegnazioni di spin agli altri quattro livelli eccitati.

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Mandelstam Representation for the Ladder Approximation of the Bethe-Salpeter Formalism.

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Summary. — A Mandelstam representation is shown to exist for each term of the ladder approximation of the scattering amplitude, in the covariant Bethe-Salpeter formalism.

1. – Introduction.

Recently, several authors ([1-3]) have proved the Mandelstam representation ([4]) for the scattering amplitude in a special case of non-relativistic potential scattering, where the potential is restricted to be a superposition of Yukawa potentials. This suggests that the same result should hold at least for the scattering amplitude obtained from the ladder approximation of the covariant Bethe-Salpeter formalism. The purpose of the present note is to show that this conjecture is true in the sense that a Mandelstam representation may be written down for each term of the ladder approximation.

For simplicity, we consider the scattering of two non-identical scalar nucleons, of equal masses m, interacting through scalar neutral mesons of mass μ. If p_1 and p_2 are the four-momenta of the incoming nucleons, p_1' and p_2' those

([1]) J. Bowcock and A. Martin: *Nuovo Cimento*, **14**, 516 (1959).

([2]) R. Blankenbecler, M. L. Goldberger, N. N. Khuri and S. B. Treiman: *Annals of Phys.*, **10**, 62 (1960).

([3]) A. Klein: *Journal of Math. Phys.*, **1**, 41 1960).

([4]) S. Mandelstam: *Phys. Rev.*, **112**, 1344 (1958).

of the outgoing nucleons, we introduce the following notations:

$$(1.1)\qquad \begin{cases} P = p_1 + p_2 = p_1' + p_2' , \\ k = \tfrac{1}{2}(p_1 - p_2) ; \qquad k' = \tfrac{1}{2}(p_1' - p_2') , \\ s = W^2 = -P^2 = 4(m^2 + k^2) = 4(m^2 + k'^2) , \\ t = \Delta^2 = \tfrac{1}{4}(k - k')^2 . \end{cases}$$

In order to get a convenient expression for the ladder approximation $T(s, t)$ of the scattering amplitude, we start from an integral representation of the ladder approximation $\varphi(x_1, x_2)$ of the Bethe-Salpeter wave function. This representation has been introduced by Okubo and Feldman [5]. One writes:

$$(1.2)\qquad \varphi(x_1, x_2) = \exp\left[i\tfrac{1}{2}P(x_1 + x_2)\right]\int (\mathrm{d}p)^4 \Psi(p) \exp\left[ip(x_1 - x_2)\right] .$$

The Fourier transform $\Psi(p)$ satisfies the Bethe-Salpeter equation:

$$(1.3)\qquad \Psi(p) = \delta(p - k) - \frac{ig^2}{(2\pi)^4}\left[(\tfrac{1}{2}P + p)^2 + m^2 - i\varepsilon\right]^{-1} \cdot \left[(\tfrac{1}{2}P - p)^2 + m^2 - i\varepsilon\right]^{-1}\int (\mathrm{d}p')^4 \left[(p - p')^2 + \mu^2 - i\varepsilon\right]^{-1}\Psi(p') .$$

The Okubo-Feldman representation reads:

$$(1.4)\qquad \Psi(p) = \delta(p - k) - \frac{g^2}{16\pi^2}\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}z\, z^2 f(x, y, z, s) \cdot \cdot \exp\left[-iz\{((\tfrac{1}{2}P + p)^2 + m^2)x + ((\tfrac{1}{2}P - p)^2 + m^2)y + (p - k)^2 + \mu^2\}\right] .$$

Okubo and Feldman show that this representation is consistent with the Bethe-Salpeter equation (1.3), and that the last is equivalent to the following integral equation for the weight-function $f(x, y, z, s)$:

$$(1.5)\qquad f(x, y, z, s) = 1 + \frac{ig^2}{16\pi^2}\int_0^x \mathrm{d}u \int_0^y \mathrm{d}v \int_0^\infty \mathrm{d}w\, \frac{f(u, v, z + w, s)}{1 + u + v} \cdot \cdot \exp\left[i\left\{\left(\frac{suv - m^2(u + v)^2}{1 + u + v} - \mu^2\right)w - \mu^2 z\left(1 + \frac{z}{w}\right)(1 + u + v)\right\}\right] .$$

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[5] S. Okubo and D. Feldman: *Phys. Rev.*, **117**, 292 (1960).

One gets the scattering amplitude $T(s, t)$ from the wave function $\varphi(x_1, x_2)$ with the help of the reduction formula:

$$T(s,t) = -\frac{i}{(2\pi)^3}\int (\mathrm{d}x)^4 \exp[-ik'x][\Box_{x/2} - m^2][\Box_{-x/2} - m^2]\varphi\left(\frac{x}{2}, -\frac{x}{2}\right). \tag{1.6}$$

Using (1.3), (1.4) and (1.5), one obtains:

$$T(s,t) = -\frac{g^2}{(2\pi)^3}\int (\mathrm{d}p')^4[(p-p')^2 + \mu^2 - i\varepsilon]^{-1}\Psi(p') = \tag{1.6}$$

$$= -\frac{ig^2}{(2\pi)^3}\int_0^\infty \mathrm{d}z\, f(z, s)\exp[-iz(4t + \mu^2)],$$

where

$$f(z, s) = \lim_{\substack{x\to\infty \\ y\to\infty}} f(x, y, z, s). \tag{1.7}$$

The last form of $T(s, t)$ in (1.6) shows that this function has an analytic continuation into the lower half of the t-plane. Furthermore, one can infer from (1.5) that $f(x, y, z, s)$, and therefore $f(z, s)$ too, has a continuation into the upper half of the s-plane. Thus: $T(s, t)$ is regular in the domain $\mathrm{Im}\, s > 0$, $\mathrm{Im}\, t < 0$. In order to get more information about the analytic properties of $T(s, t)$, we solve (1.5) by iteration. This leads to an expansion of $f(x, y, z, s)$ in powers of the square of the coupling constant g:

$$f(x, y, z, s) = \sum_{n=1}^{\infty}\left(\frac{g^2}{16\pi^2}\right)^{n-1} f_n(x, y, z, s). \tag{1.8}$$

Insertion of (1.8) into (1.6) gives

$$T(s,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{g^2}{16\pi^2}\right)^n T_n(s, t), \tag{1.9}$$

with

$$T_n(s,t) = -i\int_0^\infty \mathrm{d}z\, f_n(z, s)\exp[-iz(4t + \mu^2)], \tag{1.10}$$

$$f_n(z, s) = \lim_{\substack{x\to\infty \\ y\to\infty}} f_n(x, y, z, s). \tag{1.11}$$

In the following, we investigate the analytic properties of $T_n(s, t)$, which corresponds to a ladder-diagram with n exchanged mesons (Fig. 1). To this end we have first to set up a suitable expression of $f_n(z, s)$.

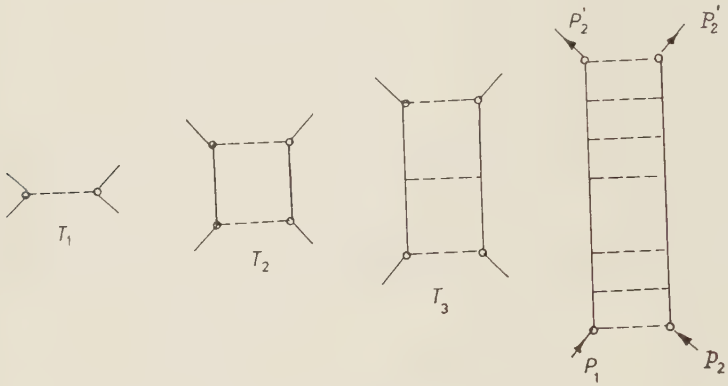

Fig. 1. – Diagrams of the ladder approximation.

2. – An expression for $f_n(z, s)$.

Straightforward application of (1.5), (1.8) and (1.11) gives

$$
\begin{aligned}
(2.1)\qquad f_n(z, s) = i^{n-1} \int_0^\infty \mathrm{d}u_1 \int_0^\infty \mathrm{d}v_1 \int_0^\infty \mathrm{d}w_1 \int_0^{u_1} \mathrm{d}u_2 \int_0^{v_1} \mathrm{d}v_2 \int_0^\infty \mathrm{d}w_2 \ldots \cdot \\
\cdot \int_0^{u_{n-2}} \mathrm{d}u_{n-1} \int_0^{v_{n-2}} \mathrm{d}v_{n-1} \int_0^\infty \mathrm{d}w_{n-1} \Big[\prod_{k=1}^{n-1} (1 + u_k + v_k)^{-1} \Big] \cdot \\
\cdot \exp \left[i \sum_{k=1}^{n-1} \left\{ \left(\frac{s u_k v_k - m^2 (u_k + v_k)^2}{1 + u_k + v_k} - \mu^2 \right) w_k - \right. \right. \\
\left. \left. - \mu^2 \left(z + \sum_{j=1}^{k-1} w_j \right) \left(1 + \frac{1}{w_k} \left(z + \sum_{j=1}^{k-1} w_j \right) \right) (1 + u_k + v_k) \right\} \right] .
\end{aligned}
$$

We transform this expression by a set of changes of variables. In a first step, we introduce variables x_k, y_k, q_k:

$$
(2.2)\qquad \left\{ \begin{array}{l} x_k = u_k + v_k \,, \\ y_k = u_k - v_k \,, \\ q_k = \dfrac{w_k}{z(1 + u_k + v_k)} \,. \end{array} \right.
$$

Choosing these as variables of integration, we get:

$$(2.3)\qquad f_n(z,s)=\left(\frac{iz}{2}\right)^{n-1}\int_0^\infty \mathrm{d}q_1\ldots\mathrm{d}q_n\int_0^\infty \mathrm{d}x_{n-1}\int_{-x_{n-1}}^{y_{n-1}}\mathrm{d}y_{n-1}\ldots\int_{x_2}^{\infty}\mathrm{d}x_1\int_{y_2-x_1+x_2}^{y_2+x_1-x_2}\mathrm{d}y_1\cdot$$

$$\cdot\exp\left[iz\sum_{k=1}^{n-1}\left\{\left(\frac{1}{4}s(x_k^2-y_k^2)-m^2x_k^2-\mu^2(1+x_k)\right)q_k-\mu^2\left(1+\sum_{j=1}^{k-1}q_j(1+x_j)\right)\cdot\right.\right.$$

$$\left.\left.\cdot\left(1+x_k+\frac{1}{q_k}\left(1+\sum_{j=1}^{k-1}q_j(1+x_j)\right)\right)\right\}\right].$$

Now, we eliminate y_k in favour of η_k defined by:

$$(2.4)\qquad y_k=\eta_{n-1}x_{n-1}+\sum_{j=k}^{n-2}\eta_j(x_j-x_{j+1})\,.$$

The limits of integration of η_k are simply -1 and $+1$. In a last step we replace the x_k's by variables x, ε_2, ..., ε_{n-1}, defined by

$$(2.5)\qquad x_1=x\,,\qquad x_2=\varepsilon_2 x\,,\qquad \ldots\,,\qquad x_{n-1}=\varepsilon_{n-1}x\,.$$

Our final expression for $f_n(z,s)$ thus reads:

$$(2.6)\qquad f_n(z,s)=\left(\frac{iz}{2}\right)^{n-1}\int_0^\infty \mathrm{d}q_1\ldots\mathrm{d}q_{n-1}\int_0^1\mathrm{d}\varepsilon_2\int_0^{\varepsilon_2}\mathrm{d}\varepsilon_3\ldots\cdot$$

$$\cdot\int_0^{\varepsilon_{n-2}}\mathrm{d}\varepsilon_{n-1}\int_{-1}^{+1}\mathrm{d}\eta_1\ldots\mathrm{d}\eta_{n-1}\left[\prod_{k=1}^{n-1}(\varepsilon_k-\varepsilon_{k+1})\right]\cdot\int_0^\infty \mathrm{d}x\,x^{2n-3}\exp\left[-iz(Ax^2+Bx+C)\right],$$

where

$$(2.7)\qquad \begin{cases} A=\sum_{k=1}^{n-1}\left[-\frac{1}{4}s\left(\varepsilon_k^2-\left(\sum_{j=k}^{n-1}\eta_j(\varepsilon_j-\varepsilon_{j+1})\right)^2\right)q_k+m^2\varepsilon_k^2q_k+\mu^2\alpha_k\left(\frac{\alpha_k}{q_k}+\varepsilon_k\right)\right], \\ B=\mu^2\sum_{k=1}^{n-1}\left[\varepsilon_kq_k+2\,\frac{\alpha_k\beta_k}{q_k}+\alpha_k+\beta_k\varepsilon_k\right], \\ C=\mu^2\sum_{k=1}^{n-1}\left[q_k+\frac{\beta_k^2}{q_k}+\beta_k\right]. \end{cases}$$

In (2.6) and (2.7), the following notations have been used:

$$(2.8)\qquad \begin{cases} \varepsilon_1=1\,, & \varepsilon_n=0\,; \\ \alpha_k=\sum_{j=1}^{k-1}q_j\varepsilon_j\,, & \alpha_1=0\,; \\ \beta_k=1+\sum_{j=1}^{k-1}q_j\,, & \beta_1=1. \end{cases}$$

3. – Analyticity of $T_n(s, t)$ as a function of the squared momentum transfer t.

In this section, we consider $T_n(s, t)$ for a fixed real value of s, such that $s < 4m^2$, and begin with t real and positive. As we know already, $T_n(s, t)$ is regular in the upper half of the s-plane. Therefore, a point $s < 4m^2$ can be reached through analytic continuation in this half-plane from the physical points $s > 4(m^2 + t)$. Now it is easy to see that, if $s < 4m^2$, the quantity A defined in (2.7), is positive. Furthermore, B is still positive, and $C > (n^2 - 1)\mu^2$. Thus $Ax^2 + Bx + C > (n^2 - 1)\mu^2$, and (2.6) shows that $f_n(z, s)$ has an analytic continuation into the lower half of the z-plane. Putting:

$$f_n(z, s) = \exp\left[-iz(n^2 - 1)\mu^2\right]\bar{f}_n(z, s) \tag{3.1}$$

we prove in Appendix A that

$$\begin{cases} |\bar{f}_n(z, s)| < C_1 |z|^{-\frac{1}{2}}, \\ |\bar{f}_n(z, s)| < C_2 |z|^{-\frac{3}{2}}, \end{cases} \tag{3.2}$$

for $\operatorname{Im} z < 0$, $s < 4m^2$ and $n > 2$. As the Mandelstam representation of $T_1(s, t)$ $\left(-1/(4t + \mu^2)\right)$ and $T_2(s, t)$ has been established by other methods [6], we shall restrict ourself in the following to the case $n > 2$.

The inequalities (3.2) show that we can shift the path of integration in (1.10) on the negative imaginary z-axis, getting:

$$T_n(s, t) = -\int_0^\infty \mathrm{d}z \exp\left[-z(4t + (n\mu)^2)\right] \bar{g}_n(z, s), \tag{3.3}$$

with

$$\bar{g}_n(z, s) = \bar{f}_n(-iz, s); \qquad \bar{g}_n^*(z, s) = \bar{g}_n(z^*, s). \tag{3.4}$$

The function $\bar{g}_n(z, s)$ is regular in the right half of the z-plane $(\operatorname{Re} z > 0)$ and has therefore a Laplace representation:

$$\bar{g}_n(z, s) = \int_0^\infty \mathrm{d}\tau \exp\left[-\tau z\right] h_n(s, \tau), \tag{3.5}$$

$$h_n(s, \tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}z \exp\left[iz\tau\right] \bar{g}_n(z, s); \qquad h_n(s, \tau) = h_n^*(s, \tau). \tag{3.6}$$

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[6] S. Mandelstam: *Phys. Rev.*, **115**, 1741 (1959). See also: J. M. Charap and S. P. Fubini: *Nuovo Cimento*, **14**, 540 (1959), and J. Tarski, to be published in *Journ. of Math. Phys.*

It follows from (3.2) that:

$$|h_n(s, \tau)| \leqslant H < \infty \,. \tag{3.7}$$

According to (3.3) and (3.5) we have

$$T_n(s, t) - T_n(s, 0) = \tag{3.8}$$

$$= \int_0^\infty \mathrm{d}z \exp\left[-z(n\mu)^2\right] \left(1 - \exp\left[-4tz\right]\right) \int_0^\infty \mathrm{d}\tau \exp\left[-\tau z\right] h_n(s, \tau) \,.$$

From (3.7) we see that we are allowed to exchange the order of integrations in the right-hand-side of (3.8); then the z-integration can be performed and we obtain:

$$T_n(s, t) = T_n(s, 0) + 4t \int_0^\infty \mathrm{d}\tau \frac{h_n(s, \tau)}{(\tau + (n\mu)^2)(4t + \tau + (n\mu)^2)} \,. \tag{3.9}$$

This expression shows that $T_n(s, t)$ is regular in the whole t-plane, except for a cut on the negative real axis starting at $t = -(n\mu)^2/4$. From (3.7), $T_n(s, t)$ is bounded by $C|t|^\alpha$, with $\alpha > \frac{1}{2}$.

The fact that we have to perform a subtraction in (3.8), before we can permute the order of the integrations over z and τ, is due to our inability of finding an upper bound of $|h_n(s, \tau)|$ going to zero as τ tends to infinity. However, as we have no reasonable doubt that $h_n(s, \tau)$ in fact vanishes at infinity, we could quite safely write down a non-subtracted version of (3.9).

4. – Analyticity of $T_n(s, t)$ as a function of the squared total energy s.

We now have to look at the properties of $f_n(z, s)$ as a function of s. If $s < 4m^2$ and $z > 0$, the path of the x-integration in (2.6) can be shifted from the positive real axis on the negative imaginary axis ($x \to -ix$). Performing further the change of variable: $A^{\frac{1}{2}}x \to x$, one gets:

$$f_n(z, s) = (-iz/2)^{n-1} \int_0^\infty \mathrm{d}q_1 \ldots \mathrm{d}q_{n-1} \int_0^1 \mathrm{d}\varepsilon_2 \ldots \int_0^{\varepsilon_{n-2}} \mathrm{d}\varepsilon_{n-1} \cdot \tag{4.1}$$

$$\cdot \int_{-1}^{+1} \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_{n-1} \left[\prod_{k=1}^{n-1} (\varepsilon_k - \varepsilon_{k+1})\right] A^{(-n-1)} \int_0^\infty \mathrm{d}x \, x^{2n-3} \exp\left[z\left(ix^2 - \frac{B}{A^{\frac{1}{2}}} x - iC\right)\right] .$$

This expression suggests that $f_n(z, s)$ is regular at any point s, such that $\mathrm{Re}\, A^{\frac{1}{2}} > 0$ for every possible value of the parameters ε_k, η_k and q_k. Comparing with (2.7), we see that this is the case when s is anywhere in the s-plane provided with a cut along the positive real axis starting from $s = 4m^2$.

We show in Appendix B that the integral in (4.1) converges when s is in the cut plane, and that $(\theta = \arg(s - 4m^2))$

$$(4.2)\quad \begin{cases} |f_n(z, s)| < C \dfrac{1}{|\sin\theta|^n} \dfrac{1}{|s-4m^2|^{(n-1)a}} \dfrac{1}{z^{(n-1)(1-a)}}, \\ \qquad\qquad \text{for } 0 < \theta < \pi/2 \text{ and } 3\pi/2 < \theta < 2\pi; \\ |f_n(z, s)| < C \dfrac{1}{|s-4m^2|^{(n-1)a}} \dfrac{1}{z^{(n-1)(1-a)}}, \\ \qquad\qquad \text{for } \pi/2 \leqslant \theta \leqslant 3\pi/2, \end{cases}$$

with $0 < a < 1$. Therefore, $f_n(z, s)$ is regular in the whole s-plane cut along $(4m^2, \infty)$. From (4.1) it is clear that the same is true for $f_n^*(z, s)$.

Equation (1.10) and the regularity of $f_n(z, s)$ in the lower half z-plane give

$$(4.3)\quad T_n(s, 0) = -i\int_0^\infty dz\, f_n(s, z) \exp[-iz\mu^2] =$$

$$= -\frac{i}{2}\int_0^\infty dz\, (f_n(z, s) \exp[-iz\mu^2] + \text{compl. conj.}).$$

Further, (3.6) reads:

$$(4.4)\quad h_n(s, \tau) = \frac{1}{2\pi}\int_0^\infty dz\, \left(f_n(z, s) \exp\left[iz(\tau + (n^2-1)\mu^2)\right] + \text{compl. conj}\right).$$

From the preceding results on $f_n(z, s)$ we infer that both $T_n(s, 0)$ and $h_n(s, \tau)$ are regular in the cut s plane, with $T_n(s^*, 0) = T_n^*(s, 0)$, $h_n(s^*, \tau) = h_n^*(s, \tau)$, and

$$|T_n(s, 0)| \leqslant F_n(s), \qquad |h_n(s, \tau)| \leqslant \frac{1}{\pi} F_n(s)$$

with

$$(4.5)\quad F_n(s) = \int_0^\infty dz\, |f_n(z, s)|.$$

Choosing, for example, $a = 1 - 1/2(n-1)$ for $0 \leqslant z \leqslant (1/m^2)$, and $a = 1 - 3/2(n-1)$ for $(1/m)^2 < z < \infty$, the inequalities (4.2) give:

$$(4.6)\quad \begin{cases} F_n(s) < \dfrac{1}{(\sin\theta)^n}\left[\dfrac{C_1}{|-s-4m^2|^{n-\frac{3}{2}}} + \dfrac{C_2}{|s-4m^2|^{n-\frac{5}{2}}}\right], \\ \qquad\qquad \text{for } 0 < \theta < \pi/2 \text{ and } 3\pi/2 < \theta < 2\pi; \\ F_n(s) < \left[\dfrac{C_1}{|s-4m^2|^{n-\frac{3}{2}}} + \dfrac{C_2}{|s-4m^2|^{n-\frac{5}{2}}}\right], \\ \qquad\qquad \text{for } \pi/2 \leqslant \theta \leqslant 3\pi/2. \end{cases}$$

Thus, $T_n(s, 0)$ and $h_n(s, \tau)$ go to zero as $(s - 4m^2)$ goes to infinity in any direction other than that of the positive real axis (remember that we consider the case $n > 2$). This implies the existence of Stieltjes representations for these functions:

$$\text{(4.7)} \qquad \begin{cases} T_n(s, 0) = \int_0^\infty \mathrm{d}\sigma \, \dfrac{\varrho_n^{(1)}(\sigma)}{s - \sigma - 4m^2} , \\ h_n(s, \tau) = \int_0^\infty \mathrm{d}\sigma \, \dfrac{\varrho_n^{(2)}(\sigma, \tau)}{s - \sigma - 4m^2} , \end{cases}$$

where $\varrho_n^{(1)}(\eta)$ and $\varrho_n^{(2)}(\sigma, \tau)$ are real. Inserting (4.7) into (3.9) we get a representation for $T_n(s, t)$ of the Mandelstam type:

$$\text{(4.8)} \qquad T_n(s, t) = \int_0^\infty \mathrm{d}\sigma \, \frac{\varrho_n^{(1)}(\sigma)}{s - \sigma - 4m^2} + \\ + 4t \int_0^\infty \mathrm{d}\tau \int_0^\infty \mathrm{d}\sigma \, \frac{\varrho_n^{(2)}(\sigma, \tau)}{[\tau + (n\mu)^2][4t + \tau + (n\mu)^2][s - \sigma - 4m^2]} .$$

This representation establishes that each term of the ladder approximation of the scattering amplitude is the boundary value of an analytic function. This function is regular in the topological product of the s-plane cut along $(4m^2, \infty)$ and the t-plane cut along $(-\frac{1}{4}(n\mu)^2, -\infty)$, n being the number of exchanged mesons.

Appendix A

An upper bound for $|f_n(z, s)|$ when $s < 4m$ and $\operatorname{Im} z < 0$.

As $f_n(-z^*, s) = f_n^*(z, s)$ $(\operatorname{Im} z < 0)$, it is sufficient to consider the case $0 > \arg z > -\pi/2$. According to (2.6), one has

$$\text{(A.1)} \qquad |f_n(z, s)| < \left| \frac{z}{2} \right|^{n-1} \int_0^\infty \mathrm{d}q_1 \ldots \mathrm{d}q_{n-1} \int_0^1 \mathrm{d}\varepsilon_2 \ldots \int_0^{\varepsilon_{n-2}} \mathrm{d}\varepsilon_{n-1} \cdot \\ \cdot \int_{-1}^{+1} \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_{n-1} \Big[\prod_{k=1}^{n-1} (\varepsilon_k - \varepsilon_{k+1}) \Big] \, M ,$$

where

$$M = \left| \int_0^\infty \mathrm{d}x\, x^{2n-3} \exp\left[-ix(Ax+B)z\right] \right| . \tag{A.2}$$

As $0 < \varphi < \pi/2$ ($\varphi = -\arg z$), we can rotate the path of integration of this integral by $\pi/6$ in the lower half plane ($x \to x \exp[-i\pi/6]$). This gives

$$M < \int_0^\infty \mathrm{d}x\, x^{2n-3} \exp\left[-x\left(Ax \sin\left(\varphi - \frac{\pi}{3}\right) + B \sin\left(\varphi + \frac{\pi}{6}\right)\right)|z|\right] < \tag{A.3}$$

$$< \int_0^\infty \mathrm{d}x\, x^{2n-3} \exp\left[-\frac{x}{2}(Ax+B)|z|\right] .$$

Inserting the last upper bound of M into (A.1) and going back to the original variables of integration x_k and y_k, we get:

$$|f_n(z,s)| < \left\{ |z| \int_0^\infty \mathrm{d}q \int_0^\infty \mathrm{d}x \int_0^x \mathrm{d}y \cdot \right. \tag{A.4}$$

$$\left. \cdot \exp\left[-\frac{|z|}{2}\left(\left(m^2 - \frac{1}{4}s\right)(x^2-y^2)q + \mu^2 x(1+q)\right)\right]\right\}^{n-1} ,$$

using the fact that

$$\begin{cases} Ax^2 > \left(m^2 - \frac{1}{4}s\right) \sum_{k=1}^{n-1} (x_k^2 - y_k^2) q_k , \\ Bx > \mu^2 \sum_{k=1}^{n-1} x_k (1+\varepsilon_k) . \end{cases} \tag{A.5}$$

We perform the integrations over q and y in (A.4) and obtain

$$|f_n(z,s)| < \left[\frac{1}{m^2 - \frac{1}{4}s} \int_0^\infty \mathrm{d}x\, N(x) \exp\left(\frac{|z|\mu^4 x}{-2(m^2 - \frac{1}{4}s)}\right)\right]^{n-1} , \tag{A.6}$$

where

$$N(x) = \frac{2}{[x(1+x)]^{\frac{1}{2}}} \log\left[(1+x)^{\frac{1}{2}} + x^{\frac{1}{2}}\right] . \tag{A.7}$$

We remark that

$$x^a N(x) \leqslant \bar{N}(a) < \infty \qquad \text{for } 0 \leqslant x < \infty, \tag{A.8}$$

as long as $0 < a < 1$. Therefore

$$|f_n(z, s)| < \left[\frac{\bar{N}(a)}{m^2 - \frac{1}{4}s}\int_0^\infty \mathrm{d}x\, x^{-a} \exp\left(\frac{z}{m^2 - \frac{1}{4}s}\right)\right]^{n-1} C(a, s)\, z^{-(n-1)(1-a)}, \tag{A.9}$$

with $C(a, s) < \infty$ for $s < 4m^2$. If $n > 2$ we can choose successively $a = 1 - 1/2(n-1)$ and $a = 1 - 3/2(n-1)$ and get the upper bounds (3.2).

Appendix B

An upper bound for $f_n(z, s)$ when s is in the cut plane and $z > 0$.

According to (4.1), we have

$$|f_n(z, s)| < \left(\frac{z}{2}\right)^{n-1} \int_0^\infty \mathrm{d}q_1 \ldots \mathrm{d}q_{n-1} \int_0^1 \mathrm{d}\varepsilon_2 \ldots \int_0^{\varepsilon_{n-1}} \mathrm{d}\varepsilon_{n-1} \cdot \int_{-1}^{+1} \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_{n-1} \left[\prod_{k=1}^{n-1} (\varepsilon_k - \varepsilon_{k+1})\right] P\, |A|^{-(n-1)}, \tag{B.1}$$

with

$$P = \left| \int_0^\infty \mathrm{d}x\, x^{2n-3} \exp\left[zx\left(ix - \frac{B}{A^{\frac{1}{2}}}\right)\right] \right|. \tag{B.2}$$

If $\pi \leqslant \arg(s - 4m^2) \leqslant 2\pi$, one has $0 \geqslant \arg A^{-\frac{1}{2}} > -\pi/2$, and we are allowed to shift the path of integration on the diagonal of the first quadrant ($x \to x \exp[i\pi/4]$). This gives

$$P < |A|^{n-1} \int_0^\infty \mathrm{d}x\, x^{2n-3} \exp[-zx(2|A|x + B)]. \tag{B.3}$$

Taking into account that

$$Ax^2 > \varrho \sum_{k=1}^{n-1} (x_k^2 - y_k^2) q_k, \tag{B.4}$$

where

$$\varrho = \begin{cases} |\frac{1}{4}s - m^2| & \text{for } \pi \leqslant \arg(s - 4m^2) \leqslant 3\pi/2, \\ |\operatorname{Im}(\frac{1}{4}s - m^2)| & \text{for } 3\pi/2 < \arg(s - 4m^2) < 2\pi, \end{cases} \tag{B.5}$$

we obtain the upper bound (4.2) by a technique similar to that used in Appendix A.

If $0 < \arg(s - 4m^2) \leqslant \pi$, one has $0 \leqslant \arg A^{-\frac{1}{2}} < \frac{1}{2} \arg(s - 4m^2)$, and the path of integration in (B.3) can be rotated by an angle $\beta = \frac{1}{2}\left((\pi/2) - \frac{1}{2} \arg(s - m^2)\right)$ in the upper half plane $(x \to x \exp[i\beta])$. After that, one procedes as before and gets the upper bound (4.2) again.

RIASSUNTO (*)

Si dimostra che esiste una rappresentazione di Mandelstam per ogni termine dell'approssimazione a gradini dell'ampiezza di scattering, nel formalismo covariante di Bethe-Salpeter.

(*) *Traduzione a cura della Redazione.*

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The Renormalization Costants in Quantum Electrodynamics (*).

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Summary. — The asymptotic behaviour at high energies of the vertex function in quantum electrodynamics is discussed with particular consideration of the gauge invariance of the theory.

1. – Introduction.

A proof of the infinity of the renormalization constants in quantum electrodynamics was given by Källén (1) without explicit use of perturbation theory. The main step in his proof can be formulated as follows. Under the assumption that the multiplicative renormalization constants $Z_1 - Z_2$ and Z_3 are finite, one establishes a formula for the asymptotic behaviour of the renormalized vertex function when the momentum transfer tends to infinity. The claim is that the limit is given by the Born approximation multiplied by Z_1

$$\lim \Gamma_\mu^{\text{ren}}(p, q) = Z_1 \gamma_\mu \,. \tag{1}$$

Here the two momenta p and q are on the mass shell, $\lim [-(p-q)^2] = \infty$, Γ_μ^{ren} is the « fully truncated » vertex function of Dyson (2). Having established the above asymptotic behaviour, Källén's proof proceeds without difficulty.

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(1) G. Källén: *Handb d. Phys.*, **5**, part I (Berlin, 1958), pp. 358-363. Reference to the original papers can be found there.

(2) F. Dyson: *Phys. Rev.*, **75**, 1736 (1949).

Alternatively, one can make use of a result due to LEHMANN, SYMANZIK and ZIMMERMANN [3], and generalized to quantum electrodynamics by EVANS [4], which states that the vertex part actually vanishes in the above limit. One concludes then immediately that

$$Z_1 = 0 \,. \tag{2}$$

Källén's derivation of (1) has been criticized recently on various counts [5]. We shall be concerned in this note only with the question of the consistency of (1) from the point of view of gauge invariance. This question was raised by JOHNSON [6], who observed that while $\Gamma^{\mathrm{ren}}_{\mu}$ is gauge invariant and directly connected with experimentally observable quantities, the renormalization constant Z_1 is not gauge invariant as one can immediately verify by calculating it to second order in perturbation theory. Johnson's remark indicates that (1) cannot be true, unless it happens accidentally to be so in a particular gauge.

In this note we shall make use of the exact formulas previously obtained [7,9] for the gauge dependence of propagators and renormalization constants. Our conclusion will be that (1) is indeed inconsistent with gauge invariance, and that it is very unlikely that it holds in any particular gauge, as for instance in the gauge used by KÄLLÉN [10]. On the other hand we shall suggest that (1) is valid provided one takes a different high energy limit. Indeed, if one keeps

[3] H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: *Nuovo Cimento*, **2**, 425 (1955). SYMANZIK has later shown that the result is valid also if subtractions are required in the propagator. A similar conclusion has been reached by S. WEINBERG.

[4] L. E. EVANS: *High Energy Theorem in Quantum Electrodynamics*, preprint.

[5] S. G. GASIOROWICZ, D. R. YENNIE and H. SUURA: *Phys. Rev. Lett.*, **2**, 513 (1959).

[6] K. JOHNSON: *Phys. Rev.*, **112**, 1367 (1958).

[7] L. D. LANDAU and I. M. KHALATNIKOV: *Žurn. Éksp. Teor. Fiz.*, **29**, 89 (1955); English translation in *Sov. Phys. Journ. Exp. Theor. Phys.*, **2**, 69 (1956). These authors had already obtained the transformation formulas given in Sect. 2 of the present note but used them to derive approximate expressions for the propagators, rather than the exact transformation formulas for the renormalization constants. We prefer to re-establish afresh the gauge formulas in order to reach a clear statement of the conditions of their validity.

[8] K. JOHNSON and B. ZUMINO: *Phys. Rev. Lett.*, **3**, 351 (1959).

[9] B. ZUMINO: *Journ. Math. Phys.*, **1**, no. 1 (1960). In formula (A.6) of this paper a normalization factor should be inserted.

[10] KÄLLÉN uses a gauge which differs from the one used in the more conventional formulation (called here the Feynman gauge). This is easy to see if one notices that, in Källén's formalism, the renormalized photon propagator is given by

$$\mathscr{D}^{\mathrm{ren}}_{\mu\nu} = (g_{\mu\nu} - 2Mk_\mu k_\nu)\frac{1}{k^2} + \int_0^\infty \left(g_{\mu\nu} + \frac{k_\mu k_\nu}{a}\right)\frac{\Pi(-a)}{a(k^2+a)}\,\mathrm{d}a = \frac{g_{\mu\nu}}{k^2} + \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\int_0^\infty \frac{\Pi(-a)}{a(k^2+a)}\,\mathrm{d}a$$

only one of the electron momenta on the mass shell, let us say p, and considers the $\lim(-q^2) = \infty$, the gauge properties of Γ_μ^{ren} are consistent with a relation like (1). In the Appendix we shall actually sketch a formal proof of this limiting statement, based on Ward's identity, when the limit is taken keeping also the photon momentum $k = p - q$ on the mass shell. In the specified limit, (1) provides a definition for Z_1 in terms of the renormalized vertex function fully analogous to the well-known definitions of Z_2, δm and Z_3 in terms of the high energy behaviour of the renormalized electron and photon propagators. (An alternative definition of Z_1 can be obtained, of course, for $p = q$ and $\lim(-p^2) = \infty$.)

It is interesting to observe that LEHMANN, SYMANZIK and ZIMMERMAN have been able to prove [11] (for pseudoscalar meson theory) the vanishing of the truncated vertex function also if one keeps on the mass shell p and k and takes the $\lim(-q^2) = \infty$. However the proof, based on the analysis of the fermion propagator, cannot be extended in a simple way to quantum electrodynamics because of the lack of positive definiteness of the contributions relating to various numbers of particles in the intermediate states. Actually, as shown in eq. (35) below, the cut-off dependence of Z_1 is very strongly affected by a change of gauge. A similar strong gauge dependence can be shown to be valid for the asymptotic behaviour of the vertex function in the limit here considered. These facts seem to indicate that in a successful proof of the inconsistency of unrenormalized quantum electrodynamics one may have to avoid the consideration of unphysical quantities such as Z_1 [12].

2. – Gauge transformations.

In the paper quoted in reference [9] we considered a class of gauge transformations which we shall call here gauge transformations of the third kind

since

$$2M = \int_0^\infty \frac{(-a)}{a^2}\,\mathrm{d}a\,.$$

If one writes the corresponding unrenormalized propagator, one can see that Källén's gauge belongs to the class considered in Section **3**, for the choice $\gamma = 1 - Z_3$. The considerations about the Feynman gauge made in Section **4** seem to apply to Källén's gauge also. For a discussion of the photon propagator, from the operator point of view, see H. FRIED: *Phys. Rev.*, **115**, 220 (1959).

[11] See reference [3]. Also E. FERRARI and G. JONA-LASINIO: *Nuovo Cimento*, **10**, 310 (1958).

[12] As an example of such an approach one may take the recent preprint by K. JOHNSON: *Quantum electrodynamics in the infinite energy limit*, which makes use of the Coulomb gauge formulation of the theory.

and which are closely related to the well-known gauge transformations of second kind [13]

$$A_\mu \to A_\mu + \partial_\mu \Lambda\,, \qquad \psi \to \psi \exp\,[ie\Lambda]\,. \tag{3}$$

It was shown there that an infinitesimal gauge transformation of third kind is given by

$$\delta Z = \frac{i}{2} \iint \frac{\delta}{\delta \Lambda(x')}\, \delta M(x' - x'')\, \frac{\delta}{\delta \Lambda(x'')}\, \mathrm{d}x'\, \mathrm{d}x''\, Z\,, \tag{4}$$

where $\delta M(x' - x'') = \delta M(x'' - x')$ is an infinitesimal function of the invariant $(x' - x'')^2$ and $Z - Z_1[\eta, \bar{\eta}, J_\mu]$ is the generating functional of propagators. Since the gauge function Λ is independent of the source functions η, $\bar{\eta}$ and J_μ, the same formula (4) is valid if one replaces Z by any propagator obtained from Z by functional differentiation with respect to the source functions. On the other hand the dependence of any propagator upon the gauge function Λ is always immediately obtained for instance from (3) and the definition of the propagator in terms of Heisenberg operators. Using (4) one can then write down explicitly the change of any propagator under a finite gauge transformation of third kind. It was shown that the gauge dependence of the zero order photon propagator

$$D_{\mu\nu} \to D_{\mu\nu} + \partial_\mu \partial_\nu M \tag{5}$$

is shared by the full photon propagator

$$\mathcal{G}_{\mu\nu} \to \mathcal{G}_{\mu\nu} + \partial_\mu \partial_\nu M\,, \tag{6}$$

while the electron propagator transforms as

$$G(x - y) \to G(x - y) \exp\,[ie^2[M(x - y) - M(0)]]\,. \tag{7}$$

For our study of the vertex functions we need the transformation law for the three point function

$$C_\mu(x - z,\ y - z) = -\langle T\, \psi(x) A_\mu(z)\, \bar{\psi}(y) \rangle \tag{8}$$

which is

$$C_\mu(x - z,\ y - z) \to \Big\{ C_\mu(x - z,\ y - z) - ie\, \frac{\partial}{\partial z_\mu}\, [M(x - z) - M(y - z)]\, G(x - y) \Big\} \cdot \exp\,\big\{ ie^2 [M(x - y) - M(0)] \big\}\,. \tag{9}$$

[13] In this section we consider only unrenormalized quantities. For instance, we denote with e the unrenormalized charge. In this paper we shall always use a superscript $^{\mathrm{ren}}$ to indicate that a quantity is renormalized.

It will be convenient to write the above formulas in momentum space. In general this cannot be done in a simple way unless one considers infinitesimal transformations δM. However we shall see below that the formulas in momentum space simplify considerably when momenta are put on the mass shell or are made to tend to infinity. In these special cases one is again able to write explicit formulas for finite transformations.

In momentum space (6), (7) and (9) become, respectively

$$\delta \mathscr{G}_{\mu\nu}(k) = -k_\mu k_\nu \delta M(k)\,, \tag{10}$$

$$\delta G(p) = \frac{ie^2}{(2\pi)^4}\int \mathrm{d}p'\,\delta M(p')[G(p-p') - G(p)] \tag{11}$$

and

$$\delta C_\mu(p,q) = \frac{ie^2}{(2\pi)^4}\int \mathrm{d}p'\,\delta M(p')[C_\mu(p-p',\, q-p') - C_\mu(p,q)] - \tag{12}$$

$$- e(p-q)\;\;\delta M(p-q)[G(q) - G(p)]\,.$$

Landau and Khalatnikov (7) observed that (12) takes a simpler form for the three point function $\widetilde{C}_\mu$ truncated by dropping all radiative corrections in the external photon line

$$C_\mu(p,q) = e\mathscr{G}_{\mu\nu}(k)\,\widetilde{C}_\nu(p,q)\,, \qquad k = p-q. \tag{13}$$

from

$$\delta C_\mu = e\mathscr{G}_{\mu\nu}\delta\widetilde{C}_\nu + e\,\delta\mathscr{G}_{\mu\nu}\widetilde{C}_\nu = e\mathscr{G}_{\mu\nu}\delta\widetilde{C}_\nu - ek_\mu k_\nu \delta M\widetilde{C}_\nu$$

and the generalized Ward identity

$$k_\nu \widetilde{C}_\nu(p,q) = G(q) - G(p) \tag{14}$$

one obtains simply

$$\delta\widetilde{C}_\mu(p,q) = \frac{ie^2}{(2\pi)^4}\int \mathrm{d}p'\,\delta M(p')[\widetilde{C}_\mu(p-p',\, q-p') - \widetilde{C}_\mu(p,q)]\,. \tag{15}$$

One can finally write explicitly the change in the vertex function

$$\Gamma_\mu(p,q) = -iG^{-1}(p)\,\widetilde{C}(p,q)\,G^{-1}(q) \tag{16}$$

as

$$\delta\Gamma_\mu(p,q) = \frac{ie^2}{(2\pi)^4}\int \mathrm{d}p'\,\delta M(p')\cdot \tag{17}$$

$$\cdot[G^{-1}(p)\,G(p-p')\,\Gamma_\mu(p-p',\, q-p')\,G(q-p')\,G^{-1}(q) +$$

$$+ \Gamma_\mu(p,q) - G^{-1}(p)\,G(p-p')\,\Gamma_\mu(p,q) - \Gamma_\mu(p,q)\,G(q-p')\,G^{-1}(q)]\,.$$

3. – Renormalization constants.

We proceed now to derive transformation formulas for the renormalization constants. As explained in reference (8), it is necessary to introduce a regularization consisting of an infrared cut-off in the form of a small photon mass in the photon propagator as well as an ultraviolet cut-off. The photon mass is necessary in order to make sure that the propagators have simple pole singularities corresponding to the one particle states. It should be pointed out that the small photon mass does not destroy the validity of the transformation formulas given above since the transformation of third kind follows from the gauge properties of the Dirac equation, which are not affected by the presence of a photon mass. (This is particularly obvious if one examines the derivation of the transformation of third kind given in the Appendix of reference (9).) Indeed, it is easy to see that a gauge transformation of third kind can be used to establish a connection between neutral vector meson theory and quantum electrodynamics regularized with a photon mass.

Now consider first (11). If the momentum p is put on the mass shell, the second term in the bracket will dominate over the first, provided the function $\delta M(p)$ is assumed to be sufficiently regular, and one obtains

$$\delta G(p) \sim -\frac{ie^2}{(2\pi)^4}\int \mathrm{d}p'\, \delta M(p')\, G(p) \tag{18}$$

or

$$G(p) \to G(p) \exp\left[-ie^2 M(0)\right] . \tag{19}$$

This gives the transformation law for the residue at the pole

$$Z_2 \to Z_2 \exp\left[-ie^2 M(0)\right] . \tag{20}$$

on the other hand take the $\lim(-p^2) = \infty$. We shall always take for $\delta M(p')$ a function of $(p')^2$, vanishing sufficiently rapidly for large values of the argument. Writing

$$G(p) = f(p^2) + i\gamma p g(p^2) , \tag{21}$$

we have in the specified limit

$$G(p - p') \sim f(p^2) + i\gamma p g(p^2) - i\gamma p' g(p^2) = G(p) - i\gamma p' g(p^2) . \tag{22}$$

If one inserts (22) into (11), one obtains

$$\delta G(p) \sim 0 , \tag{23}$$

in agreement with the statement that, at infinite $-p^2$,

$$\lim (i\gamma p + m) G(p) = 1 . \tag{24}$$

Similar considerations can be made for (17). If both p and q are on the mass shell only the second term in the bracket survives

$$\delta \Gamma_\mu(p, q) = \frac{ie^2}{(2\pi)^4} \int \mathrm{d}p' \, \delta M(p') \, \Gamma_\mu(p, q) \tag{25}$$

or

$$\Gamma_\mu(p, q) \to \Gamma_\mu(p, q) \exp [ie^2 M(0)] . \tag{26}$$

Setting now $p = q$ one obtains, from

$$\Gamma_\mu(p, p) = \frac{1}{Z_1} \gamma_\mu , \tag{27}$$

the transformation formula

$$Z_1 \to Z_1 \exp [- ie^2 M(0)] . \tag{28}$$

We see that (20) and (28) are identical, in agreement with the equality of Z_1 and Z_2. This equality is preserved when one goes to a different gauge. However we also see that once both p and q are on the mass shell (26) is valid even if $p \neq q$, in particular if $\lim [- (p - q)^2] = \infty$. Therefore a statement that in this limit

$$\lim \Gamma_\mu(p, q) = \gamma_\mu \tag{29}$$

is inconsistent with gauge invariance (clearly (29) when expressed in terms of the renormalized vertex function becomes identical with (1)).

On the other hand, set only p on the mass shell. Now in (17) one is left with the second and the fourth term in the bracket. If in addition $-q^2$ is very large, one can see that the second and fourth term cancel and one has in this limit

$$\delta \Gamma_\mu(p, q) \sim 0 , \tag{30}$$

in agreement with (29). Alternatively, one could set $q = p$ and take the

$\lim(-p^2) \to \infty$. From (17) one would still deduce an equation like (30) and would be led to conjecture the validity of

$$\lim \Gamma_\mu(p, p) = \gamma_\mu \,. \tag{31}$$

Actually (31) follows directly from Ward's identity

$$i\Gamma_\mu(p, p) = \frac{\partial}{\partial p_\mu} G^{-1}(p) \tag{32}$$

if one assumes the correct asymptotic behaviour for $G(p)$

$$\lim (i\gamma p + m) G(p) = 1 \,. \tag{33}$$

Similarly, one can use the generalized Ward identity (14) to give a convincing argument for the validity of (29). This is done briefly in the Appendix.

One should observe that (20) and (28) do not represent just the multiplication by a phase factor, so that the vanishing of Z_1 is by no means an obviously gauge invariant statement. For instance, one usually considers the class of gauges

$$D_{\mu\nu} = \left(g_{\mu\nu} - \gamma \frac{k_\mu k_\nu}{k^2}\right)\left(\frac{1}{k^2 + \mu^2} - \frac{1}{k^2 + \Lambda^2}\right), \tag{34}$$

where γ is a real constant. For $\gamma = 0$ one obtains the Feynman gauge, regularized by the small photon mass μ and by the ultraviolet cut-off Λ. For $\gamma \neq 0$ one goes into other gauges and it is easily seen that (28) takes the form [8]

$$Z_1 \to Z_1 \exp\left[\gamma \frac{e^2}{8\pi^2} \log \frac{\Lambda}{\mu}\right]. \tag{35}$$

The multiplicative factor is in this case real. If one remembers the second order Feynman gauge expression

$$Z_1 = 1 - \frac{e^2}{8\pi^2}\left(\log \frac{\Lambda}{m} + \frac{9}{4} - 2 \log \frac{m}{\mu}\right) \tag{36}$$

(m is the electron mass), one obtains the well-known result that to this order the choice $\gamma = 1$ cancels the ultraviolet divergence, while the choice $\gamma = -2$ cancels the infrared divergence of Z_1.

From (26) and (28) we see that the renormalized vertex function is invariant

with respect to a gauge transformation of third kind provided the two momenta are on the mass shell. This invariance is a particular example of the general invariance of all matrix elements of the renormalized collision matrix. The general proof could be given in complete analogy to the special case. When the momenta of the external lines are put on the mass shell, the unrenormalized matrix elements transform by multiplicative gauge factors. Exactly these factors are absorbed by the transformation of the renormalization constants.

4. – Comparison with perturbation theory.

The verification of formulas such as (1) or (29) in perturbation theory requires some care, due to the divergence of the renormalization constants. The safest procedure seems to be the use of the Pauli-Villars regularization method ([14]). In addition to the cut-offs appearing in (34), one introduces a gauge invariant regularization of the closed loops. It is then easy to see that all the gauge transformation formulas are valid for the regularized theory. Statements about the asymptotic behaviour of propagators at high energies must then be verified by letting the appropriate variable tend to infinity while all the cut-off parameters are kept fixed at some very large value ([15]) (compared with m). (One should also notice that the regularization procedure does not affect the analiticity properties of form factors except for the addition of very distant singularities, so that dispersion formulas of the usual type will still be valid. The weight functions will now depend upon the cut-off parameters and will ensure stronger convergence of the dispersion integrals.)

One can now proceed to check the asymptotic formulas. This will not be done here explicitly. It is easy to convince oneself that perturbation theory supports the statements made in the previous sections. On the other hand, if one tries to check eq. (29), for the unrenormalized vertex function when p and q are on the mass shell and $\lim [-(p-q)^2] \to \infty$, one finds agreement only to order e^2, in the Feynman gauge. In higher orders there does not appear to be agreement. For instance, if one takes the simplest vertex diagram with three vertices, but one uses in it a photon propagator including higher order corrections, one obtains a term proportional to γ_μ, which does not vanish in the limit. This term arises from the part of the photon propagator proportional to $k_\mu k_\nu$ (see reference ([10])). It cannot be excluded, of course that there exists a gauge in which (29) is true in the specified limit.

([14]) W. Pauli and F. Villars: *Rev. Mod. Phys.*, **21**, 434 (1949).
([15]) See K. Symanzik: *Nuovo Cimento*, **11**, 269 (1959).

* * *

In conclusion the author wishes to thank Dr. K. Symanzik for many illuminating conversations on the subjects treated in this note. He also would like to thank Professor J. R. Oppenheimer for the hospitality afforded him by the Institute for Advanced Study.

Appendix

In this Appendix, the momentum p will be always taken on the mass shell. Using the formulas given in the text it is easy to verify in this case that

$$X_\mu(p, q) = \Gamma_\mu(p, q)\, G(q)(i\gamma q + m) \tag{A.1}$$

is gauge invariant. Clearly one can also write

$$X_\mu(p, q) = \Gamma_\mu^{\text{ren}}(p, q)\, G^{\text{ren}}(q)(i\gamma q + m)\,. \tag{A.2}$$

We want to show that, as $\lim(-q^2) = \infty$ and k is on the mass shell,

$$\lim X_\mu(p, q) = \gamma_\mu. \tag{A.3}$$

This equation then implies (1) or (29).

The generalized Ward identity can be written as

$$-ik_\mu X_\mu = i\gamma q + m \tag{A.4}$$

or, with

$$X_\mu = \gamma_\mu + \Xi_\mu, \tag{A.5}$$

as

$$k_\mu \Xi_\mu = 0\,. \tag{A.6}$$

This relation, together with relativistic invariance implies the structure

$$\begin{aligned}\Xi_\mu = {} & i\sigma_{\mu\nu}k_\nu[F_1(-q^2) - F_2(-q^2)i\gamma q] + k_\mu[mF_3(-q^2) + q^2F_4(-q^2) - \\ & - \{mF_4(-q^2) - F_3(-q^2)\}i\gamma q] + i\gamma_\mu k^2[F_3(-q^2) - F_4(-q^2)i\gamma q]\,.\end{aligned} \tag{A.7}$$

Here the form factors are functions of k^2 as well as of $-q^2$

$$F_i(-q^2) = F_i(-q^2,\ k^2)\,. \tag{A.8}$$

Setting now k on the mass shell, the terms in (A.7) containing F_3 and F_4 vanish. The term with F_1 and F_2 will vanish also in the $\lim(-q^2) = \infty$, if one assumes a reasonable behaviour at infinity for the form factors, such as is implied, for instance by the dispersion relations

$$F_i(-q^2) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{\operatorname{Im} F_i(\varkappa^2)}{q^2 + \varkappa^2 - i\varepsilon}\, \mathrm{d}\varkappa^2 \,. \tag{A.8}$$

It is clear, on the other hand, that the generalized Ward identity does not give much information when both p and q are on the mass shell.

RIASSUNTO (*)

Si discute il comportamento asintotico della funzione di vertice alle alte energie in elettrodinamica quantistica tenendo particolarmente in considerazione l'invarianza di gauge della teoria.

(*) *Traduzione a cura della Redazione.*

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Dispersion Relations for Scattering Wave Functions in Potential Theory.

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Summary. — Using the well known analytic properties of partial wave functions, dispersion relations are derived which can be transformed by unitarity into singular integral equations. In absence of bound-states a procedure is given which allows to determine, order by order, the wave function when the S-matrix is known at every order. A quite intuitive deduction of dispersion relations for the case of finite range potential based on the completeness relation is also reported. This result allows the derivation of dispersion relations for the transition matrix elements.

1. – Introduction.

Recently many authors [1,3] have investigated the analytic properties of the scattering amplitude for the Schrödinger equation with respect to both variables; energy and momentum transfer. Those studies succeded in giving the proof for potential theory of the representation first proposed by Mandelstam [4] for field theory.

Furthermore the Mandelstam representation has been used in order to obtain the scattering matrix using only the dispersion relations and unitarity.

[1] J. Bowcock and A. Martin: *Nuovo Cimento*, **14**, 516 (1959).
[2] T. Regge: *Nuovo Cimento*, **14**, 951, (1959).
[3] R. Blankenbecler, M. L. Goldberger, N. N. Khuri and S. B. Treiman: *Ann. Phys.* (to be published).
[4] S. Mandelstam: *Phys. Rev.*, **112**, 1344 (1958).

This approach to the theory of the scattering matrix has been discussed by GELL-MANN (5).

In this paper we want to derive a dispersion relation for the scattering wave function ... in terms of the energy E keeping x fixed. This result will enable us to discuss whether the program of Gell-Mann can be applied also to the determination of the wave function from the scattering matrix using analyticity and unitarity.

For this purpose, using well known analytic properties of the wave function we shall derive a dispersion relation which by unitarity can be transformed into a singular integral equation.

Such an integral equation has an infinity of possible solutions. However, adding to this equation requirements of asymptotic behaviour and of analyticity in the coupling constant it is possible to obtain a unique determination of the wave function. This of course, coincides with the one that could be obtained by a direct solution of the Schrödinger equation. The situation has some analogy to the one met in the case of the Low equation where the requirement of analyticity in g^2 is sufficient to eliminate the ambiguity found by CASTILLEJO, DALITZ and DYSON (6).

In Section **2** we begin to derive the dispersion relation for the finite range potential.

The derivation presented there, is very simple and intuitive and it is based on the completeness relations for the wave function. A rigorous proof can be found in Appendix A.

In Section **3** we generalize the result to the infinite range potential.

In Section **4** we transform the dispersion relation into a singular integral equation and discuss the relative solution. Finally we will formulate a prescription in order to derive order by order the wave function and apply such a prescription to a very simple example.

2. – Dispersion relation for finite range potential.

We will limit ourselves to the S-waves.

In the following we shall choose units so that the Schrödinger equation will be written as follows

$$\psi'' + (k^2 - V(x))\,\psi = 0\,, \tag{1}$$

where $V(x) = 0$ for $x \geqslant a$.

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(5) M. GELL-MANN: *High Energy Nuclear Physics: Proceedings of the Sixth Annual Rochester Conference* (1956), III p. 30. This program has been applied to the potential theory in reference (3) and to quantize theories by S. MANDELSTAM.

(6) L. CASTILLEJO, R. W. DALITZ and F. J. DYSON: *Phys. Rev.*, **101**, 453 (1956).

We will consider the solutions $\varphi^{\text{out}}(k, x)$ and $\varphi^{\text{in}}(k, x)$ of eq. (1) defined by their asymptotic behaviour for large x *viz.*:

$$(2) \qquad \begin{cases} \varphi^{\text{out}}(k, x) \sim \dfrac{\exp[-ikx] - S \exp[ikx]}{-2ik} \\ \varphi^{\text{in}}(k, x) \sim \dfrac{\exp[ikx] - S^* \exp[-ikx]}{2ik} \end{cases} \qquad \text{for } x \geqslant a$$

and

$$S = \exp[2i\delta] .$$

These functions satisfy the relation

$$(3) \qquad \varphi^{\text{out}}(k, x) = S\varphi^{\text{in}}(k, x) .$$

From (3) and from the reality of $V(x)$ it follows that

$$(4) \qquad \begin{cases} \varphi^{\text{out}}(k, x) = \varrho(k, x) \exp[i\delta] , \\ \varphi^{\text{in}}(k, x) = \varrho(k, x) \exp[-i\delta] , \end{cases}$$

with $\varrho(k, x)$ real function.

In the following we shall frequently refer to eq. (4) as unitarity statements.

We also note that from time reversal one gets:

$$(5) \qquad \varphi^{\text{out}}(-k, x) = \varphi^{\text{in}}(k, x) .$$

It is worth while to observe that (4) and (5) are also valid in the general case when the potential is not of finite range. Besides these states belonging to the continuum spectrum we suppose the existence of m bound states with binding energies $-\chi_\varrho^2$ and with eigenfunctions $\sqrt{N_\varrho}\psi_\varrho(x)$.

The asymptotic behaviour of these functions is:

$$(6) \qquad \sqrt{N_\varrho}\psi_\varrho(x) \sim \sqrt{N_\varrho} \exp[-\chi_\varrho x]$$

and

$$N_\varrho^{-1} = \int_0^\infty \psi_\varrho^2(x)\, \mathrm{d}x .$$

For such a spectrum the completeness relation reads:

$$(7) \qquad \frac{2}{\pi}\int_0^\infty k^2\, \mathrm{d}k \varphi^{\text{out}}(k, x)\, \varphi^{*\text{out}}(k, x') + \sum_{\varrho=1}^{m} N_\varrho \psi_\varrho(x)\, \psi_\varrho(x') = \delta(x - x') .$$

Let now x be arbitrary and $x' \geqslant a$; then we may replace in (7) the function $\varphi^{\text{out}}(k, x')$ and $\psi_\varrho(x')$ with their asymptotic expressions (2) and (6) and we get:

$$\frac{1}{i\pi}\int_0^\infty \mathrm{d}k\, k\, \varphi^{\text{out}}(k, x)\{\exp[ikx'] - S^*(k)\exp[-ikx']\} + \\ + \sum_{\varrho=1}^{m} N_\varrho \Psi_\varrho(x) \exp[-\chi_\varrho x'] = \delta(x - x') . \tag{8}$$

Now from (3) one gets $\varphi^{\text{out}}(k, x)\, S^* = \varphi^{\text{in}}(k, x)$ and (8) can be rewritten:

$$\frac{1}{\pi i}\int_0^\infty k\, \mathrm{d}k\{\varphi^{\text{out}}(k, x)\exp[ikx'] - \varphi^{\text{in}}(k, x)\exp[-ikx']\} + \\ + \sum_{\varrho=1}^{m} N_\varrho \psi_\varrho(x) \exp[-\chi_\varrho x'] = \delta(x - x') . \tag{8'}$$

Introducing the Fourier transform of $\delta(x - x')$ and using eq. (5) we get finally:

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left\{k\,\varphi^{\text{out}}(k, x) - \frac{i}{2}\exp[-ikx]\right\}\exp[ikx']\,\mathrm{d}k + \\ + i\sum_{\varrho=1}^{m} N_\varrho \psi_\varrho(x)\exp[-\chi_\varrho x'] = 0 , \tag{9}$$

that must be valid for every $x' \geqslant a$.

From this relation it is quite easy to derive formally a dispersion relation.

Indeed let us introduce the function

$$\chi(k, x) = \left\{k\,\varphi^{\text{out}}(k, x) - \frac{i}{2}\exp[-ikx]\right\}, \tag{10}$$

then eq. (9) reads:

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\chi(k, x)\exp[ikx']\,\mathrm{d}k + i\sum_{\varrho=1}^{m} N_\varrho \psi_\varrho(x)\exp[-\chi_\varrho x'] = 0 \qquad \text{for } x' \geqslant a . \tag{9'}$$

We can now transform the integration path into a semicircle Γ of infinite radius in the upper half plane plus small circles on the imaginary axis avoiding the singularities coming from the bound states.

Then from (9') we obtain

$$\frac{i}{2\pi i}\sum_\varrho \oint_{i\chi_\varrho} \chi(k, x)\exp[ikx']\,\mathrm{d}k - \frac{1}{2\pi i}\int_\Gamma \chi(k, x)\exp[ikx']\,\mathrm{d}k = \\ = -\sum_{\nu=1}^{m}\frac{N_\nu}{2}\psi_\nu(x)\exp[-\chi_\nu x'] . \tag{11}$$

Since this equation must be valid for every $x' \geqslant a$ the coefficient of $\exp[ikx']$ on both sides must be equal and we get:

$$\frac{1}{2\pi i}\oint_{i\chi_\varrho} \chi(k, x)\,\mathrm{d}k = -\frac{N_\varrho}{2}\psi_\varrho(x)\int_\Gamma \chi(k, x)\,\mathrm{d}k = 0\,. \tag{12}$$

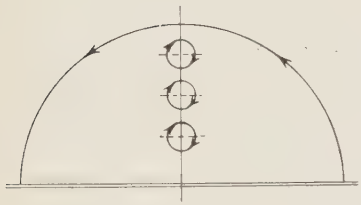

Fig. 1.

We can now evaluate $\chi(k, x)\exp[ika]$ by the Cauchy theorem:

$$\chi(k, x)\exp[ika] = \frac{1}{2\pi i}\oint \frac{\chi(k', x)\exp[ik'a]}{k' - k}\,\mathrm{d}k'\,, \tag{13}$$

where the path of integration is the same as in Fig. 1. By virtue of (12) we write:

$$\mathrm{Re}\{\chi(k, x)\exp[ika]\} = \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{Im}\{\chi(k', x)\exp[ik'a]\}}{k' - k}\,\mathrm{d}k' - \tag{14}$$

$$-2\,\mathrm{Re}\,\frac{1}{2\pi i}\sum_{\varrho=1}^{m}\oint\frac{\chi(k', x)\exp[ik'a]}{k' - k}\,\mathrm{d}k'\,.$$

From (10) follows

$$\begin{cases} \mathrm{Re}\{\chi(k, x)\exp[ika]\} = k\,\mathrm{Re}\{\varphi^{\mathrm{out}}(k, x)\exp[ika]\} + \frac{1}{2}\sin k(a - x) \\ \mathrm{Im}\{\chi(k, x)\exp[ika]\} = k\,\mathrm{Im}\{\varphi^{\mathrm{out}}(k, x)\exp[ika]\} - \frac{1}{2}\cos k(a - x) \end{cases} \tag{15}$$

Furthermore by (4) and (5) we write

$$\mathrm{Re}\{\varphi^{\mathrm{out}}(k, x)\exp[ika]\} + \frac{1}{2}\frac{\sin k(a - x)}{k} = \tag{16}$$

$$-\frac{1}{\pi}\int_0^\infty\frac{\mathrm{Im}\{\varphi^{\mathrm{out}}(k', x)\exp[ik'a]\}\mathrm{d}k'^2}{k'^2 - k^2} - \frac{1}{\pi}\int_0^\infty\frac{\cos k'(a - x)\,\mathrm{d}k'^2}{k'^2 - k^2} -$$

$$-\frac{2}{\kappa}\,\mathrm{Re}\left\{\frac{1}{2\pi i}\sum_{\varrho=1}^{m}\oint\frac{\chi(k', x)\exp[ik'a]}{k' - k}\,\mathrm{d}k'\right\}.$$

Since the poles of $\chi(k', x)$ on the imaginary axis are simple we have:

$$\frac{1}{2\pi i}\oint_{i\chi_\varrho}\frac{\exp[ik'a]\,\chi(k, x)\,\mathrm{d}k'}{k' - k} = \frac{\exp[-\chi_\varrho a]}{i\chi_\varrho - k}\,\frac{1}{2\pi i}\oint\frac{\chi(k', x)\,\mathrm{d}k'}{k' - k}\,,$$

and by (12):

$$\frac{1}{2\pi i}\oint_{i\chi_\varrho}\frac{\exp[ik'a]\,\chi(k',x)\,dk'}{k'-k}=-\frac{N_\varrho}{2}\frac{\exp[-\chi_\varrho a]}{i\chi_\varrho-k}\psi_\varrho(x)\,. \tag{17}$$

Finally recalling that

$$P\int_0^\infty (a^2-x^2)^{-1}\cos(xy)\,dx=\frac{\pi}{2}\,a^{-1}\sin ay \qquad \text{if } y>0$$

the equation (16) reads (7):

$$\begin{aligned}\mathrm{Re}\{\varphi^{\mathrm{out}}(k,x)\exp[ika]\}&=\sum_{\varrho=1}^{m}N_\varrho\frac{\exp[-\chi_\varrho a]\psi_\varrho(x)}{k^2+\chi_\varrho^2}\\&+\frac{\sin k(x-a)}{k}\,\eta(x-a)+\frac{1}{\pi}P\int\frac{\mathrm{Im}\{\varphi^{\mathrm{out}}(k',x)\exp[ik'a]\}dk'^2}{k'^2-k^2}\,,\end{aligned} \tag{18}$$

and

$$\eta(x-a)\begin{cases}1 & \text{if } x\geqslant a\\ 0 & \text{if } x<a\,.\end{cases}$$

As is easily seen, the dispersion relation (18) can be utilized in order to get dispersion relations for the transition matrix element.

This problem will be discussed in a next paper.

3. – Dispersion relations for infinite range potential.

We shall generalize here the previous results to the case of an infinite range potential. For this aim we use the general properties of the Jost functions (8).

We introduce the functions $\Phi(k,x)$ and $f(k,x)$ which are solutions of the Schrödinger equation with the following boundary conditions:

$$\begin{cases} a) & \Phi(k,0)=0\,, \qquad \Phi'(k,0)=1\,,\\ b) & \lim_{x\to\infty}\exp[ikx]f(k,x)=1\,. \end{cases} \tag{19'}$$

(7) A more refined analysis shows that our formula (18) is valid provided a means twice the range of the potential (see Appendix A).

(8) R. Jost and W. Kohn: *Dan. Mat. Fys. Medd.*, **27**, 9 (1953).

It is known that $\Phi(k, x)$ is an entire function of k. For large $|k|$

$$\Phi(k, x) \sim \frac{\sin kx}{k} . \tag{19''}$$

On the other hand $f(k, x)$ is regular for $\mathrm{Im}\,(k) < 0$ and continuous for $\mathrm{Im}\,(k) \leqslant 0$. For large $|k|$ and $\mathrm{Im}\,(k) \leqslant 0$

$$f(k, x) \sim \exp\,[-ikx] . \tag{19'''}$$

We recall also the properties of the function

$$f(k) \equiv f(k, 0) . \tag{20}$$

It can be shown that $\lim_{x\to\infty} f(k) \to 1$ on the real axis and along any direction in the lower half plane.

From the general theory of the scattering matrix we know that the zeros of $f(k)$ correspond to bound states if they lie on the lower imaginary axis and the binding energies are given by

$$E_\varrho = -\chi_\varrho^2 ; \qquad f(-i\chi_\varrho) = 0 \qquad \chi_\varrho > 0 . \tag{21}$$

For real k one has:

$$\Phi(k, x) = \frac{1}{2ik}\,[f(k) f(-k, x) - f(-k) f(k, x)] . \tag{22}$$

Recalling that the scattering matrix $S(k)$ is given by $S(k) = f(k)/f(-k)$ one sees at once that

$$\varphi^{\mathrm{out}}(k, x) = \frac{\Phi(k, x)}{f(-k)} . \tag{23}$$

We may conclude that $\Phi(k, x)/f(-k)$ is analytic in the whole upper complex plane if one avoids the points $i\chi_\varrho$ (defined by eqs. (21)) which are simple poles corresponding to bound states.

Moreover from the asymptotic behaviour of the functions $\Phi(k, x)$ and $f(-k)$ it follows that the function

$$\chi(k, x) = \varphi^{\mathrm{out}}(k, x) \exp\,[ikx] = \frac{\Phi(k, x)}{f(-k)} \exp\,[ikx] \tag{24}$$

is for large $|k|$ and $\mathrm{Im}\,(k) \geqslant 0$

$$\chi(k, x) = O\left(\frac{1}{k}\right) . \tag{25}$$

We can therefore assert that

$$\chi(k, x) = \frac{1}{2\pi i}\oint \frac{\chi(k', x)}{k' - k}\,\mathrm{d}k' , \tag{26}$$

if the path is as in the Fig. 1.

We can now enlarge the path, without any contribution from the circle of infinite radius and we get remembering eqs. (4) and (5),

$$\mathrm{Re}\,\chi(k, x) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\,\chi(k', x)}{k'^2 - k^2}\,\mathrm{d}k'^2 - \mathrm{Re}\sum_{\varrho=1}^{m}\frac{1}{\pi i}\oint_{i\chi_\varrho}\frac{\chi(k', x)}{k' - k}\,\mathrm{d}k' . \tag{27}$$

Now if we call $\psi_\varrho(x)$ the eigenfunction corresponding to the bound state $-\chi_\varrho^2$ we have by means of definitions (22), (23) and (24)

$$\frac{1}{2\pi i}\oint_{i\chi_\varrho}\frac{\chi(k', x)\,\mathrm{d}k'}{k' - k} = \frac{1}{2\pi i}\oint_{i\chi_\varrho}\frac{\exp[ik'x]\,\mathrm{d}k'}{2ik'(k' - k)}\left[\frac{f(k')}{f(-k')}f(-k', x) - f(k', x)\right].$$

The second term in the bracket does not contribute to the integral and we are left with:

$$\frac{1}{2\pi i}\oint_{i\chi_\varrho}\frac{\chi(k', x)}{k' - k}\,\mathrm{d}k' = \frac{1}{2\pi i}\oint_{i\chi_\varrho}\frac{\exp[ik'x]\,\mathrm{d}k'}{2ik'(k' - k)}\,\frac{f(k')}{f(-k')}f(-k', x) .$$

Since there is only a pole of $f(-k')$ in $i\chi_\varrho$ we get:

$$\frac{1}{2\pi i}\oint\frac{\chi(k', x)}{k' - k}\,\mathrm{d}k' = \exp[-\chi_\varrho x]\,\frac{f(-i\chi_\varrho, x)f(i\chi_\varrho)}{-2\chi_\varrho(i\chi_\varrho - k)\dot{f}(-i\chi_\varrho)} , \tag{28}$$

where we have put

$$\left[\frac{\mathrm{d}f(k)}{\mathrm{d}k}\right]_{k=-i\chi_\varrho} = \dot{f}(-i\chi_\varrho) .$$

As we shall show in Appendix B the expression on the r.h.s. of (28) is

$$N_\varrho\frac{\exp[-\chi_\varrho x]\psi_\varrho(x)}{2i\chi_\varrho(i\chi_\varrho - k)} .$$

Therefore we can finally write:

$$\mathrm{Re}\left\{\varphi^{\mathrm{out}}(k, x)\exp[ikx]\right\} = \frac{1}{\pi}P\int_0^\infty\frac{\mathrm{Im}\left\{\varphi^{\mathrm{out}}(k', x)\exp[ik'x]\right\}}{k'^2 - k^2}\,\mathrm{d}k'^2 + \tag{29}$$

$$+\sum_{\varrho=1}^{m}N_\varrho\frac{\exp[-\chi_\varrho x]\psi_\varrho(x)}{k^2 + \chi_\varrho^2} .$$

4. – Singular integral equation.

In this Section we wish to discuss an interesting property of the dispersion relation obtained, that is the possibility of transforming such a relation into a singular integral equation. We consider for example the equation (29) and observe that, due to the unitarity requirements expressed by (4), we have:

$$(30)\qquad \begin{cases} \mathrm{Im}\{\varphi^{\mathrm{out}}(k, x)\exp[ikx]\} = \varrho(k, x)\sin\bar{\delta}(k, x)\,, \\ \mathrm{Re}\{\varphi^{\mathrm{out}}(k, x)\exp[ikx]\} = \varrho(k, x)\cos\bar{\delta}(k, x)\,, \end{cases}$$

where $\bar{\delta}(k, x) = \delta(k) + kx$.

Therefore we get at once the singular integral equation for $\varrho(k, x)$

$$(31)\qquad \varrho(k, x)\cos\bar{\delta}(k, x) = \sum_{\varrho=1}^{m} N_\varrho \frac{\exp[-\chi_\varrho x]\psi_\varrho}{\chi_\varrho^2 + k^2} + \frac{1}{\pi} P\int_0^\infty \frac{\varrho(k', x)\sin\bar{\delta}(k')\,\mathrm{d}k'^2}{k'^2 - k^2}\,.$$

The general theory ([9]) of the singular integral equation tells us that such an equation has not unique so'ution since we can always add a solution of the homogeneous equation:

$$(32)\qquad \varrho(k, x)\cos\bar{\delta}(k, x) = \frac{2}{\pi} P\int_0^\infty \frac{\varrho(k', x)\sin\bar{\delta}(k', x)\,\mathrm{d}k'^2}{k'^2 - k^2}$$

to the particular of equation (31).

In the following we will limit ourselves to the case where no bound-states are present ([10]) and we analyse the conclusions which can be drawn from equation (32).

Let us introduce the function of complex variable z ([11])

$$(33)\qquad \psi(z) = \frac{1}{2\pi i}\int_0^\infty \frac{\varrho(k', x)\sin\bar{\delta}(k', x)\,\mathrm{d}k'^2}{k'^2 - z}$$

([9]) N. I. Muskhelishvili: *Singular Integral Equations* (Groningen, 1953).

([10]) It is well known that a potential can give rise to bound-states only if

$$\int_0^\infty V(x)x\,\mathrm{d}x \geqslant 1\,.$$

See R. Jost and A. Pais: *Phys. Rev.*, **82**, 840 (1951).

([11]) N. I. Muskhelishvili: *Singular Integral Equations* (Groningen, 1953).

from which we obtain ($k^2 = E$)

$$\frac{1}{\pi} P \int_0^\infty \frac{\varrho(E', x) \sin \bar{\delta}(E', x)}{E' - E} = i\{\psi(E+) - \psi(E-)\}, \tag{34}$$

$$\varrho(E, x) \sin \bar{\delta}(E, x) = \psi(E+) - \psi(E-) . \tag{35}$$

Substitution of (34) in (32) leads to:

$$\varrho(E, x) \cos \bar{\delta} = i\{\psi(E+) + \psi(E-)\}. \tag{36}$$

Now from eq. (35) and (36) one can draw

$$\left| \begin{array}{ll} \varrho(E, x) \exp [i\bar{\delta}] & = 2i\psi(E+) , \\ \varrho(E, x) \exp [-i\bar{\delta}] & = 2i\psi(E-) \end{array} \right. \tag{37}$$

or

$$\varrho(E, x) = 2i\psi(E+) \exp [-i\bar{\delta}] = 2i\psi(E-) \exp [i\bar{\delta}] . \tag{38}$$

Therefore the problem is reduced to the determination of a function $\psi(z)$ which satisfies the equation:

$$\psi(E+) \exp [-i(\delta(E) + \sqrt{E}r)] = \psi(E-) \exp [i(\delta(E) + \sqrt{E}r)] . \tag{39}$$

Using standard procedure we state

$$\psi(z) = T(z) \Omega(z) , \tag{40}$$

where the function $\Omega(z)$ is defined by the condition:

$$\exp [-2i\delta(E)] \Omega(E+) - \Omega(E-) = 0 . \tag{41}$$

This admits a solution [12]

$$\Omega(z) = \exp \left[\frac{1}{\pi} \int_0^\infty \frac{\delta(E')}{E' - z} \mathrm{d}E' \right], \tag{42}$$

and $T(z)$ must satisfy the equation:

$$T(E+) \exp [-i\sqrt{E}x] = T(E-) \exp [\sqrt{E}x] . \tag{43}$$

[12] R. Omnès: *Nuovo Cimento*, **8**, 316 (1958).

But this relation shows that $T(z)\exp[-i\sqrt{z}\,x]$ is an analytic function in the whole complex plane except eventually the point 0 and ∞ where it may have singularities.

We can exclude the singularity at the origin since we know that the wave function is regular for $E=0$.

It can therefore be concluded that $T(z)\exp[-i\sqrt{z}\,x]$ is an entire function of z, $I(z,x)$:

$$T(z,x)=I(z,x)\exp[i\sqrt{z}\,x]\,.$$

From eqs. (44), (42), (40) and (37) we get:

$$\varphi^{\text{out}}(E,x)=\Omega(E+)\cdot 2iI(E,x)\,. \tag{45}$$

At this point, if one recalls the explicit expression of the Jost function $f(-k)$ ([13]) one realizes at once that eq. (45) is just the statement of the properties from which we started in order to obtain the dispersion relation since

$$\Omega(E+)=\exp\left[\frac{1}{\pi}P\int_0^\infty\frac{\delta(E')}{E'-E}\,\mathrm{d}E'+i\,\delta(E)\right]=f^{-1}(-k)\,.$$

Therefore we conclude that the analyticity and unitarity do not allow by themselves the determination of the wave function. This is not at all a surprising result, since we know that other conditions must be added in order to determine a quantity satisfying a dispersion relation.

5. - Gell-Mann program for wave functions.

Now we will see that it is possible to formulate a program for the determination of the wave function when the S-matrix is known. Also here we concentrate ourselves to the case of no bound-states. In such a case it is well known that the wave function is an analytic function of the coupling constant g ([14]).

Let us denote by $a_n(k,x)$ and $b_n(k,x)$ the real and the imaginary part of n-th order term of the wave function, then equation (29) can be written for

([13]) See for example the paper by R. Jost and W. Kohn quoted in ref. ([8]) footnote ([6]).

([14]) See the paper by R. Jost and A. Pais quoted in reference ([10]).

the n-th order:

$$a_n(E, x)\cos\sqrt{E}\,x = \frac{1}{\pi}\,P\int_0^\infty \frac{a_n(E', x)\sin\sqrt{E'}\,x}{E' - E}\,\mathrm{d}E' - g_n(E, x)\,, \tag{46}$$

where

$$g_n(E, x) = \frac{1}{\pi}\,P\int_0^\infty \frac{b_n(E', x)\cos\sqrt{E'}\,x}{E' - E}\,\mathrm{d}E' + b_n(E, x)\sin\sqrt{E}\,x\,.$$

This is equivalent to say that when the imaginary part of the n-th order term is known, the real part is the solution of the inhomogeneous singular integral eq. (46).

As far as the imaginary part $b_n(E, x)$ is concerned it is very easy to see that it is completely determined if we know the wave function until the $(n-1)$-th order and the D-matrix until the n-th order.

Therefore let us expand both sides of eq. (3) in power series of g; we have:

$$\varphi_0^{(\mathrm{out})} + g\,\varphi_1^{(\mathrm{out})} + \ldots g^n\,\varphi_n^{(\mathrm{out})} = (1 + S_1 g + \ldots S_n g^n)(\varphi_0^{(\mathrm{in})} + \varphi_1^{(\mathrm{in})} g + \ldots)$$

from which one gets:

$$\begin{cases} \varphi_0^{(\mathrm{out})} = \varphi_0^{(\mathrm{in})} = \dfrac{\sin kx}{k}\,, \\ \varphi_1^{(\mathrm{out})} = \varphi_1^{(\mathrm{in})} + S_1\,\varphi_0^{(\mathrm{in})}\,, \\ \varphi_2^{(\mathrm{out})} = \varphi_2^{(\mathrm{in})} + S_1\,\varphi_1^{(\mathrm{in})} + S_2\,\varphi_0^{(\mathrm{in})}\,, \end{cases} \tag{47}$$

or

$$\begin{cases} b_1 = \dfrac{S_1\,\varphi_0^{\mathrm{in}}}{2i}\,, \\ b_2 = \dfrac{1}{2i}\left\{S_1\,\varphi_1^{(\mathrm{in})} + S_2\,\varphi_0^{(\mathrm{in})}\right\}, \end{cases} \tag{47'}$$

and so on....

Further remarks are needed concerning the solution of the equation (46). Since we know that it is always possible to add a solution of a homogeneous equation to the particular solution of the inhomogeneous equation; we must add an additional condition in order to determine the solution univocally.

Such a further condition will be the asymptotical behaviour of the n-th order term with respect to the energy.

The asymptotical behaviour of the n-th order term can be obtained by

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Summary. — In this work the critical equation together with the stationary thermal neutron distribution is stated for a cylindrical neutron-multiplying structure. The fissionable material is assumed to be lumped into several blocks; allowance is made for an arbitrary distribution of them in the system. The source-sink technique is used throughout, while the neutron migration is dealt with by age-diffusion or multigroup approximations. Leakage and non-leakage probabilities as a function of the neutron birth position are carefully computed. A method is worked out for the simultaneous evaluation of the criticality, together with the maxima and minima of any function of the critical mass itself.

Introduction.

Many efforts have been made during the recent years in order to extend the heterogeneous methods of calculation to the finite multiplying structures.

The rather academic case of a nuclear reactor consisting of a finite number of thin fuel plates parallel to each other and separated by slabs of pure moderating material has been discussed in full detail [1]. No minor attention has been given to the square or rectangular cross-section reactors, made with axially infinite fuel rods, arranged on a square lattice [2].

In this paper a *general treatment for the cylindrical heterogeneous reactor of finite radius* is given.

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The theory has been worked out for the bare systems only, for two main reasons: firstly, if the reflector has the same nuclear properties as the moderator and does not extend to infinity, this treatment is the exact one; secondly, we note that detailed heterogeneous theories of this type will prove useful when dealing with rather compact, low weight and high power reactors, where, for shielding purposes, strong surrounding absorbers are required.

Under these conditions a reactor behaves essentially as the bare one.

In this work the first section deals primarily with the thermalization and diffusion kernels in cylindrical geometry, whilst the second part is mainly concerned with leakage and non-leakage probabilities, as a function of the source position in the moderating cylinder, for fast and thermal neutrons as well. The principles of the heterogeneous reactor theory are outlined in part three, together with their application to multiplying structures made of point-, or line-multipying blocks.

It is shown how extrema of any function—the total weight and cost, for instance—of the critical mass can be found, together with the criticality conditions.

1. – Thermalization and diffusion kernels for the bare cylinder.

1·1. *Age kernel.* – To begin with, let us consider an axially infinite, bare cylinder of pure moderating material.

A fast neutron source—stationary in time—is releasing isotropically one neutron per unit time at the space point $\boldsymbol{r} = \boldsymbol{r}_k$.

Suppose for the moment that the age theory might be safely assumed to give a good description of the thermalization process.

[In the following contest it will be clearly shown that the theory worked out in this paper is suitable for a multigroup approach as well.]

The Fermi-age equation

$$\nabla^2_{\boldsymbol{r}'} Q(\boldsymbol{r}'\,;\,\boldsymbol{r}_k\,;\,\tau) = \frac{\partial Q(\boldsymbol{r}'\,;\,\boldsymbol{r}_k\,;\,\tau)}{\partial \tau} \tag{1.1}$$

has to be solved in the given cylindrical volume for all the values of the age τ in the interval $0 < \tau \leqslant \tau_{\text{th}}$, and under the additional boundary condition

$$Q(R,\,\theta',\,z'\,;\,r_k,\,\theta_k,\,z_k\,;\,\tau) = 0\,. \tag{1.2}$$

Here the vector $\boldsymbol{r}'$ is expressed by means of its components r', θ', z' in a cylindrical frame of reference and the points $r' = R$ lie on the external surface of the moderating cylinder.

In equation (1.1) Q represents the slowing down density and $Q(\boldsymbol{r}'\,;\,\boldsymbol{r}_k\,;\,\tau)\,\mathrm{d}\boldsymbol{r}'$

Taking into account eq. (59) and (55) we get

$$(62)\qquad a_1(E,x)\sin\sqrt{E}x =$$

$$= \frac{\exp[i\sqrt{E}x]}{2\pi i}\left\{P\int_0^\infty \frac{g_1(E',x)\sin\sqrt{E'}x}{E'-E}\,dE' + i\pi g_1(E,x)\sin\sqrt{E}x\right\} -$$

$$- \frac{\exp[-i\sqrt{E}x]}{2\pi i}\left\{P\int_0^\infty \frac{g_1(E',x)\sin\sqrt{E'}x}{E'-E}\,dE' - i\pi g_1(E,x)\sin\sqrt{E}x\right\},$$

or

$$(63)\qquad a_1(E,x) - \frac{1}{\pi}P\int_0^\infty \frac{g_1(E',x)\sin\sqrt{E'}x}{E'-E}\,dE' + g_1(E,x)\cos\sqrt{E}x\,.$$

Using explicit expression for $g_1(E,x)$, we get for $a_1(E,x)$

$$(64)\quad a_1(E,x) - 2\big(1-\exp[-ux]\big)\frac{\cos\sqrt{E}x}{u(u^2+4E)} - \frac{(1-\exp[-ux])\exp[-(u/2)x]}{u(u^2+4E)}\,.$$

It is very easy to verify that this is a solution of equation (52). To (64) we can add every solution of the homogeneous equation

$$(65)\qquad a_1^0(E,x)\cos\sqrt{E}x = \frac{1}{\pi}P\int_0^\infty \frac{a_1^0(E',x)\sin\sqrt{E'}x}{E'-E}\,dE'\,.$$

Such a solution may still be defined by (53), (59). Now we obtain:

$$(66)\qquad T_0(E+) - T_0(E-) = 0\,.$$

This equation tells us that $T_0(z)$ is analytic in the whole complex plane except eventually points 0 and ∞. Also there we can exclude the singularity at $E=0$ and therefore $T_0(E,x)$ must be an entire function of E. Denoting such a function by $I(E,x)$ we get from (59) and (55) that $a_0(E,x)$ too must an be entire function. Such a function must now be determined from the asymptotical condition. From Appendix C we know that for $E\to\infty$ $\varphi_1^{(\mathrm{out})}$ behaves as:

$$(67)\qquad \operatorname{Re}\varphi_1^{(\mathrm{out})}(E,x) \underset{E\to\infty}{\sim} A(x)\frac{\cos\sqrt{E}x}{E} + B(x)\frac{\sin\sqrt{E}x}{E^{\frac{3}{2}}}\,.$$

Recalling (64) we can write:

$$\text{(68)} \qquad \operatorname{Re}\varphi_1^{(\text{out})}(E, x) = \frac{2\cos\sqrt{E}x}{u(u^2+4E)} - \frac{\exp[-ux]\cos\sqrt{E}x}{u(u^2+4E)} - \frac{(1-\exp[-ux])\exp[-(u/2)x]}{u(u^2+4E)} + \text{entire function}\,.$$

The entire function must be determined so that eq. (62) asymptotically behaves as (67).

The most general entire function satisfying such a condition is:

$$\text{(69)} \qquad I(E, x) = \frac{(1-\exp[-(u/2)]x)\exp[(-u/2)x]}{u(u^2+4E)} + \frac{C(x)\,(\sin\sqrt{E}x/\sqrt{E}) + D(x)\cos\sqrt{E}x}{u^2+4E}\,,$$

if $C(x)$ and $D(x)$ are determined so that the pole at $E = -u^2/4$ disappears.

One gets at once:

$$\text{(70)} \qquad D(x) = 0\,, \qquad C(x) = -\exp[-ux]\,.$$

Therefore φ_1^{out} will be:

$$\text{(71)} \qquad \varphi_1^{\text{out}} = \frac{2\exp[i\sqrt{E}\,x]}{u(u^2+4E)} - \frac{2\exp[-ux]\cos\sqrt{E}\,x}{u(u^2+4E)} - \frac{\exp[-ux]\sin\sqrt{E}x}{\sqrt{E}(u^2+4E)}\,,$$

which is the expression one gets solving the Schrödinger equation to the first order.

7. – Conclusions.

In the previous Sections we described a procedure allowing us to obtain the partial Schrödinger wave function at given order when the S-matrix is given until the same order, starting from the dispersion relation which follows from the analytical properties of the wave function.

We recall here briefly the whole recipe:

1) If the wave function is known until the order $(n-1)$-th and the S-matrix until the order n-th then unitarity determines the imaginary part of the n-th order wave function.

2) Once the imaginary part of the n-th order term is known the real part is a solution of an inhomogeneous singular equation (see eq. (46)).

3) The high energy behaviour of the n-th order wave function determines completely the solution of the corresponding homogeneous equation.

Therefore we can conclude that when the S matrix is known at every order the wave function can be determined order by order by such a procedure.

We have not yet considered the modifications introduced by the bound sates.

They will be discussed in a further paper.

* * *

I am deeply indebted to Prof. S. Fubini for his constant suggestions and criticism and interest in this work. Many points have been clarified discussing with him.

I want also to thank Prof. M. Verde for the very useful criticism and for suggesting to me the method of getting the aymptotic behaviour of wave functions at every order.

I have benefited of discussions with many people among whom I like to remember Prof. E. Corinaldesi and Dr. T. Regge. Finally I wish to thank Dr. E. Predazzi for the help he gave me in some points of the work.

Part of this work has been carried out during my stay at the University of Padua, and I want to thank Prof. N. Dallaporta for the hospitality extended to me.

Appendix A

We shall give here a rigorous proof of eq. (18). It is based on an investigation of T. Regge on the analyticity of the S-matrix ([16]). He was able to show that if the potential is such that $V(x) = 0$ for $x > a$ then in the upper k plane the principal term in the asymptotic series for $f(k)$ is:

$$f(k) \sim C \frac{\exp[-2ika]\Gamma(\lambda+1)}{(2ik)^{\lambda+2}}, \tag{A.1}$$

where C is a constant and λ a quantity greater than zero. Furthermore in such a case the wave function $f(k, x)$ differs from its Born approximation $\exp[-ikx]$

([16]) T. Regge: *Nuovo Cimento*, **8**, 671 (1958).

by a quantity $\Delta f(k, x)$ whose modulus is

$$|\Delta f(k, x)| < \frac{M}{h^{\lambda+2-\varepsilon}} \exp[2\beta a - \beta x] \exp\left[\frac{(a-x)V}{h}\right], \tag{A.2}$$

here $\beta = \operatorname{Im} k$, $h = |k|$, M is a constant. Let us now consider the function:

$$\chi(k, x) = \left\{k\varphi^{\text{out}}(k, x) + \frac{i}{2}\exp[-ikx]\right\}\exp[2ika].$$

If we remember the definition of $\varphi^{\text{out}}(k, x)$ and take into account (A.1) and (A.2) we see that in the upper half plane it is:

$$|\chi(k, x)| = O\left(\frac{1}{k}\right)$$

and therefore we can apply the Cauchy theorem, with a path of Fig. 1 in order to avoid the singularity. Residues can be evaluated with the results of Appendix B, and eq. (18) follows.

It is clear from the previous argument that if one does not subtract the term $(i/2)\exp[-ikx]$, in order to get the proper boundary behaviour one is obliged to replace $\exp[2ika]$ by $\exp[ikx]$. In this case one obtains eq. (29).

Appendix B

For sake of completeness we recall in this Appendix some well known results which are needed in order to justify eq. (28).

The relations we need are:

$$\psi_\varrho(x) = \frac{f(k_\varrho, x)}{f'(k_\varrho, 0)}, \qquad k_\varrho = i\chi_\varrho, \ \chi_\varrho > 0, \tag{B.1}$$

$$N_\varrho^{-1} = \int_0^\infty \psi_\varrho^2(x)\,dx = -\frac{\dot{f}(k_\varrho, 0)}{k_\varrho f'(k_\varrho, 0)}, \tag{B.2}$$

where f' e $\dot{f}$ stay for $\partial f/\partial x$ and $\partial f/\partial k$ respectively.

Relation (B.1) results from the extension of equation (22) to complex values of k, if one takes into account that $f(k_\varrho, 0) = 0$ and the identity:

$$f(k, x)f'(-k, x) - f'(k, x)f(-k, x) = 2ik.$$

In order to derive eq. (22) we integrate the quantity $f^2(k, x)(\partial/\partial k)(f''(k, x)/f(k, x))$ over x from 0 to ∞. Taking into account that this is a solution of the Schrödinger equation we get:

$$-2k\int_0^\infty f^2(k, x)\,\mathrm{d}x = \int_0^\infty f^2(k, x)\,\frac{\partial}{\partial k}\left(\frac{f''(k, x)}{f(k, x)}\right)\mathrm{d}x\,.$$

The r.h.s. can be integrated by parts: if further we put $k = k_\varrho$ and make use of (B.1) the relation (B.2) follows.

Appendix C

We must give the recursion formula for the derivation of the quantities $H_m^{(n)}$, $K_m^{(n)}$, $T_m^{(n)}$, $U_m^{(n)}$ which have been introduced in eq. (49).

It can be shown [15] that a solution of the equation:

$$\psi''(x) + (E - V(x))\,\psi(x) = 0 \tag{C.1}$$

can be written as:

$$\psi(E, x) = \Psi_0(E, x) + H\Psi_0(E, x) - K\Psi_0'(E, x)\,,$$

where Ψ_0 is the free solution which satisfies the same boundary conditions as ψ and $H(E, x)$ and $K(E, x)$ satisfy the following system of differential equations:

$$\begin{cases} E\,K'(E, x) - \dfrac{V}{2} = -\dfrac{1}{2}\left(H''(E, x) - V(x)H(E, x)\right), \\ H'(E, x) = \dfrac{1}{2}\left(K''(E, x) - V\,K(E, x)\right). \end{cases} \tag{C.2}$$

If we expand all the quantities in power series of g writing:

$$\psi = \sum_{n=0}^{\infty} g^n\psi_n\,,$$

$$H = \sum_{n=0}^{\infty} g^n H^{(n)}, \qquad K = \sum_{n=0}^{\infty} g^n K^{(n)}, \qquad \text{with } K^{(0)} = H^{(0)} \equiv 0\,,$$

we get

$$\psi_n = H^{(n)}\Psi_0 - K^{(n)}\Psi_0'(E, x)\,, \tag{C.3}$$

while $H^{(n)}$ and $K^{(n)}$ satisfy the system:

$$\text{(C.4')} \qquad \begin{cases} E\,K^{(1)\prime} - \dfrac{V}{2} = -\dfrac{1}{2}H^{(1)\prime\prime} \\ H^{(1)\prime} = \dfrac{1}{2}K^{(1)} \end{cases}$$

and for $n > 1$

$$\text{(C.4'')} \qquad \begin{cases} E\,K^{(n)\prime} = -\frac{1}{2}\left(H^{(n)\prime\prime} - V(x)\,H^{(n-1)}\right), \\ H^{(n)\prime} = \frac{1}{2}\left(K^{(n)\prime\prime} - V(x)\,K^{(n-1)}\right). \end{cases}$$

A formal solution of this system may be given writing:

$$\text{(C.5)} \qquad K^{(n)}(E, x) = \sum_{m=0}^{\infty} \frac{K_m^{(n)}(x)}{E^{m+1}} \qquad H^{(n)}(E, x) = \sum_{m=0}^{\infty} \frac{H_m^{(n)}}{E^{m+1}}.$$

Then it follows:

$$\text{(C.6)} \qquad \begin{cases} \text{for } n = 1 \\ \qquad K_0^{(1)\prime} = \dfrac{V}{2}, \qquad K_{m+1}^{(1)\prime} = -\dfrac{1}{2}H_m^{(1)\prime\prime} \\ \qquad\qquad\qquad H_m^{(1)\prime} = \dfrac{1}{2}K_m^{(1)\prime\prime} \\ \text{and for } n > 1 \\ \qquad K_{m+1}^{(n)\prime} = -\dfrac{1}{2}(H_m^{(n)\prime\prime} - V\,H_m^{(n-1)}), \\ \qquad H_m^{(n)\prime} = \dfrac{1}{2}(K_m^{(n)\prime\prime} - V\,K_m^{(n-1)}). \end{cases}$$

To each particular solution of equation (C.1) corresponds one boundary condition for the determination of $H_m^{(n)}$ and $K_m^{(n)}$. Now we go to the solutions of particular interest in our problem. Let us consider the function $\Phi(E, x)$ defined by the boundary conditions:

$$\text{(C.7)} \qquad \Phi(E, 0) = 0, \qquad \Phi'(E, 0) = 1.$$

Then

$$\Phi_0(E, x) = \frac{\sin\sqrt{E}\,x}{\sqrt{E}},$$

and the boundary conditions are in this case:

$$\text{(C.7')} \qquad \begin{cases} K_m^n(0) = 0 , \\ H_m^n(0) = K_m^{(n)\prime}(0) . \end{cases}$$

Let us now consider the solution $f(-k, x)$ which is defined by

$$\lim_{x\to\infty} f(-k, x) \exp[-ikx] = 1 .$$

This time we will write

$$\text{(C.8)} \qquad f(-k, x) = T(E, x) f_0(-k, x) - U(E, x) f_0'(-k, x)$$

in order to avoid confusion.

The quantities $T_m^{(n)}(x)$ and $U_m^{(n)}(x)$ must satisfy the boundary conditions

$$\text{(C.9)} \qquad \lim_{x\to\infty} T_m^{(n)}(x) = 0 , \qquad \lim_{x\to\infty} U_m^{(n)}(x) = 0 .$$

We now apply these expression to the determination of the asymptotic behaviour of $\Psi_1^{\text{out}}(k, x)$.

From (C.5) and (C.7) we get:

$$K_0^{(1)} \quad \frac{1}{2}\int_0^x V(x)\,\mathrm{d}x ,$$

$$H_0^{(1)} \quad \tfrac{1}{4} V(x) + \tfrac{1}{4} V(0) ,$$

which are different from zero

$$T_0^{(1)} = -\tfrac{1}{2}\int_x^\infty V(x)\,\mathrm{d}x ,$$

$$U_0^{(1)} = \tfrac{1}{2} V(x) .$$

which also are different from zero.

We have therefore (until the first order)

$$\Phi(E, x) = \frac{\sin\sqrt{E}x}{\sqrt{E}} + \frac{g}{E}\left(H_0^{(1)}\frac{\sin\sqrt{E}x}{\sqrt{E}} - K_1^{(1)}\cos\sqrt{E}x\right),$$

$$f(-k) = 1 + g\{T_0^{(1)}(0) - iU_0^{(1)}\sqrt{E}\}\frac{1}{E} .$$

Therefore

$$\lim_{E\to\infty} \varphi_1^{\text{out}}(E, x) - \left\{T_0^{(1)}(0) - i U_0^{(1)}\sqrt{E}\right\} \frac{\sin\sqrt{E}x}{E^{\frac{3}{2}}} - H_0^{(1)} \frac{\sin\sqrt{E}x}{E^{\frac{3}{2}}} - K_0^{(1)} \frac{\cos\sqrt{E}x}{E},$$

and the expression (67) follows.

RIASSUNTO

Facendo uso delle note proprietà analitiche delle funzioni d'onda parziali vengono derivate relazioni di dispersione che possono essere trasformate in equazioni integrali singolari mediante la unitarietà. In assenza di stati legati, viene descritto un procedimento atto a determinare, ordine per ordine, la funzione d'onda quando la matrice S è nota a tutti gli ordini. È anche descritto un metodo intuitivo per dedurre le relazioni di dispersione nel caso di un potenziale a range finito. Questo risultato permette una deduzione delle relazioni di dispersione per gli elementi di matrice di transizione.

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In equation (1.1) Q represents the slowing down density and $Q(\boldsymbol{r}'; \boldsymbol{r}_k; \tau)\,\mathrm{d}\boldsymbol{r}'$

gives the probability that a neutron born in $\boldsymbol{r}_k$ with age zero will reach the age τ in $\mathrm{d}\boldsymbol{r}'$ near $\boldsymbol{r}'$.

To solve (1.1) we assume Q to be the sum of two functions Q_{sing} and Q_{reg}. $Q_{\mathrm{s.ng}}(\boldsymbol{r}'; \boldsymbol{r}_k; \tau)$ is the singular solution of the age equation for the infinite medium, when only a unit point source is present at $\boldsymbol{r}_k$: so its integrable singularity for $\tau \to 0$ lies at $\boldsymbol{r} = \boldsymbol{r}_k$. $Q_{\mathrm{reg}}(\boldsymbol{r}'; \boldsymbol{r}_k; \tau)$, on the other hand, is a solution of eq. (1.1) regular in all the cylinder's volume and such that

$$Q = Q_{\mathrm{reg}} + Q_{\mathrm{sing}} \tag{1.3}$$

satisfies the boundary condition (1.2) on the external surface of the cylinder.

We assume for Q_{sing} the well known expression

$$Q_{\mathrm{sing}}(\boldsymbol{r}'; \boldsymbol{r}_k; \tau) = \frac{1}{\sqrt{(4\pi\tau)^3}} \cdot \exp\left[-\frac{\varrho^2 + (z' - z_k)^2}{4\tau}\right], \tag{1.4}$$

where $\varrho^2 = (x' - x_k)^2 + (y' - y_k)^2 = r'^2 + r_k^2 - 2r'r_k \cos(\theta' - \theta_k)$.

The Laplace transform $\bar{Q}_{\mathrm{sing}}(\boldsymbol{r}'; \boldsymbol{r}_k; p)$ of Q_{sing} with respect to τ is given by

$$\mathscr{L}_\tau[Q_{\mathrm{sing}}(\boldsymbol{r}'; \boldsymbol{r}_k; \tau)] = \frac{\exp\left[-\sqrt{p}\cdot\sqrt{\varrho^2 + (z' - z_k)^2}\right]}{4\pi\sqrt{\varrho^2 + (z' - z_k)^2}} \equiv \bar{Q}_{\mathrm{sing}}(p)\,. \tag{1.5}$$

Due to the particular geometry of the problem, the functional dependence of $\bar{Q}_{\mathrm{sing}}(p)$ on $(z' - z_k)$ may be—at least formally—simplified, by expressing the function (1.5) as an inverse Fourier's cosine transform with respect to $(z' - z_k)$. One easily gets:

$$\bar{Q}_{\mathrm{sing}}(p) = \frac{1}{2\pi^2} \int_0^\infty \cos\zeta(z' - z_k) \cdot K_0(\xi\varrho) \cdot \mathrm{d}\zeta\,, \tag{1.6}$$

being $\xi = \sqrt{\zeta^2 + p}$, and K_0 the zero order modified Bessel function of the second kind.

Equation (1.6), which gives formally $\bar{Q}_{\mathrm{s.ng}}(p)$ as an integrated product of two functions, one depending only on $(z' - z_k)$, and the other on ϱ, possesses clearly an even parity with respect to $(z' - z_k)$, as is required by (1.5). To express $Q_{\mathrm{sing}}(p)$ as a function of the polar co-ordinates r and θ, one has to remember the following addition formulas for K_0:

$$\begin{cases} K_0(\xi\varrho) = \sum_{-\infty}^{+\infty}{}_m \cos m(\theta' - \theta_k) \cdot I_m(\xi r') \cdot K_m(\xi r_k) & \text{for } r' < r_k, \quad \text{and} \\ K_0(\xi\varrho) = \sum_{-\infty}^{+\infty}{}_m \cos m(\theta' - \theta_k) \cdot K_m(\xi r') \cdot I_m(\xi r_k) & \text{for } r' > r_k. \end{cases} \tag{1.7}$$

The I_m's are the modified Bessel functions of the first kind.

From (1.6) and (1.7) one can deduce immediately that, for $r' > r_k$

$$\bar{Q}_{\text{sing}}(p) = \frac{1}{2\pi^2} \sum_{-\infty}^{+\infty}{}_m \cos m(\theta' - \theta_k) \cdot \int_0^\infty \cos \zeta(z' - z_k) \cdot I_m(\xi r_k) \cdot K_m(\xi r') \, \mathrm{d}\zeta . \tag{1.8}$$

Looking for the regular function $Q_{\text{reg}}(\boldsymbol{r}'; \tau)$ one can now choose it between all the inverse Laplace transforms of the spatially regular solutions of the equation

$$\left(\frac{\partial^2}{\partial r'^2} + \frac{1}{r'} \frac{\partial}{\partial r'} + \frac{1}{r'^2} \frac{\partial^2}{\partial \theta'^2} + \frac{\partial^2}{\partial z'^2} - p \right) \bar{Q}_{\text{reg}}(\boldsymbol{r}'; p) = 0 , \tag{1.9}$$

for which the functional dependence on the axial coordinate z' is the same as for $\bar{Q}_{\text{sing}}(p)$ in eq. (1.8).

Therefore one can take

$$\bar{Q}_{\text{reg}}(\boldsymbol{r}'; p) = \frac{1}{2\pi^2} \cdot \sum_{-\infty}^{+\infty}{}_m \cos m(\theta' - \theta_k) \cdot \int_0^\infty \cos \zeta(z' - z_k) \cdot C_m(\xi) \cdot I_m(\xi r') \, \mathrm{d}\zeta,$$

where the factors C_m's do not depend on r', but may depend on ζ and are to be determined by satisfying the boundary condition (1.2), or the equivalent one for $\bar{Q} = \bar{Q}_{\text{sing}} + Q_{\text{reg}}$. It easily follows from the orthogonality and the completeness of the trigonometric system of functions $\cos m(\theta' - \theta_k)$ in $0 \leqslant \theta' < 2\pi$ with respect to any even function that for each m

$$C_m(\xi) = - \frac{I_m(\xi r_k) \cdot K_m(\xi R)}{I_m(\xi R)} . \tag{1.10}$$

This value for C_m causes $\bar{Q}_{\text{reg}}$ to be symmetrical with respect to the interchange of $\boldsymbol{r}'$ and $\boldsymbol{r}_k$.

From (1.3) and (1.10) it follows for the L-transform of the total slowing down density in the region $\boldsymbol{r}' < \boldsymbol{r}_k$:

$$\bar{Q}(\boldsymbol{r}'; \boldsymbol{r}_k; p) = \frac{1}{(2\pi)^2} \sum_{-\infty}^{+\infty}{}_m \cos m(\theta' - \theta_k) \cdot \cdot \int_{-\infty}^{+\infty} \exp [i\zeta(z' - z_k)] \frac{I_m(\xi R) \cdot K_m(\xi r_k) - K_m(\xi R) I_m(\xi r_k)}{I_m(\xi R)} I_m(\xi r') \, \mathrm{d}\zeta. \tag{1.11}$$

For $r' > r_k$ the radii r' and r_k have to be interchanged in the above formula.

To perform the integration with respect to ζ in (1.11) it is suitable to apply

the theorem of residues to the analytic function

$$(1.12) \qquad -\frac{\exp\left[i\zeta(z'-z_k)\right] K_m(\xi R)\cdot I_m(\xi r_k)\cdot I_m(\xi r')}{\exp\left[-im(\pi/2)\right]\cdot J_m(i\xi R)}$$

of the complex variable $\zeta = \zeta_1 + i\zeta_2$. Simple poles are present in the points ζ for which $i\xi R = \alpha_{mj}$; here α_{mj} stands for the j-th root of the trascendental equation $J_m(x) = 0$. In the poles ζ_{mj} the following equalities hold:

$$i\sqrt{\zeta^2 + p}\, R = \alpha_{mj}\,; \qquad \zeta_{mj} = i\sqrt{\frac{\alpha_{mj}^2}{R^2} + p}\,.$$

For $z'-z_k > 0$ (for $z'-z_k < 0$) we choose as integration path—not passing through any pole of the integrand—the ζ_1 axis, plus a large semicircle in the upper (lower) half plane.

If we write $\sum_{j=1}^{j=\infty}$ to denote the summation over all the poles, that is the roots of $J_m(i\xi R) = 0$, we easily get as a final result

$$(1.14) \qquad \bar{Q}(\boldsymbol{r}'; \boldsymbol{r}_k; p) =$$

$$= \frac{1}{2\pi R^2} \sum_{-\infty}^{\infty} m \cos m(\theta' - \theta_k)\cdot \sum_{1}^{\infty} j \frac{\exp\left[-|z'-z_k|\sqrt{\frac{\alpha_{mj}^2}{R^2} + p}\right] J_m\left(\frac{\alpha_{mj}}{R} r'\right)\cdot J_m\left(\frac{\alpha_{mj}}{R}\cdot r_k\right)}{\sqrt{\frac{\alpha_{mj}^2}{R^2} + p}\,[J'_m(\alpha_{mj})]^2}\,.$$

The inverse Laplace transformation is immediately carried out on (1.14) by applying the following properties

$$(a_-) \qquad \mathscr{L}^{-1}\{\bar{f}(p+\alpha)\} = \exp[-\alpha\tau]\,\mathscr{L}^{-1}\{\bar{f}(p)\} = \exp[-\alpha\tau]\cdot f(\tau)\,;$$

$$(b_-) \qquad \mathscr{L}^{-1}\left[\frac{1}{\sqrt{p}}\cdot\exp\left[-|z'-z_k|\cdot\sqrt{p}\right]\right] = \frac{1}{\sqrt{\pi\tau}}\cdot\exp\left[-\frac{(z'-z_k)^2}{4\tau}\right];$$

as well as its linearity. The result is the following

$$(1.15) \qquad Q(\boldsymbol{r}'; \boldsymbol{r}_k; \tau) = \frac{\exp\left[-\frac{(z'-z_k)^2}{4\tau}\right]}{\sqrt{4\pi\tau}}\cdot$$

$$\cdot\frac{1}{\pi R^2}\cdot\sum_{-\infty}^{+\infty} m \cos m(\theta' - \theta_k)\cdot\sum_{1}^{\infty} j \frac{\exp\left[-\frac{\alpha_{mj}^2}{R^2}\tau\right] J_m\left(\frac{\alpha_{mj}}{R} r'\right) J_m\left(\frac{\alpha_{mj}}{R} r_k\right)}{[J'_m(\alpha_{mj})]^2}\,.$$

A check suggests itself at this stage: due to the continuous thermalization process, which constitutes the basic underlaying assumption of the age theory, the slowing down density $Q(\boldsymbol{r}'; \boldsymbol{r}_k; \tau)$, as τ approaches zero, is expected to take significant values only in smaller and smaller regions surrounding the source. In other words, an initial condition of this type has to be valid:

$$\lim_{\tau \to 0} Q(\boldsymbol{r}'; \boldsymbol{r}_k; \tau) = \delta(\boldsymbol{r}' - \boldsymbol{r}_k) \equiv \delta(z' - z_k) \cdot \frac{1}{r_k} \cdot \delta(r' - r_k) \cdot \delta(\theta - \theta_k) \,. \tag{1.16}$$

Now eq. (1.16) is immediately verified with respect to the axial coordinate z. Referring to the angular and radial dependence, we remark that the following formal expansion in Bessel-Fourier's series holds in the plane geometry:

$$\frac{1}{r_k} \cdot \delta(r' - r_k) \cdot \delta(\theta' - \theta_k) = \tag{1.17}$$

$$= \frac{1}{\pi R^2} \cdot \sum_{-\infty}^{+\infty}{}_m \cos m(\theta' - \theta_k) \cdot \sum_{1}^{\infty}{}_j \frac{J_m\left(\frac{\alpha_{mi}}{R} r'\right) \cdot J_m\left(\frac{\alpha_{mj}}{R} r_k\right)}{[J'_m(\alpha_{mj})]^2}$$

thus providing us with a complete proof of the eq. (1.16) in the limit $\tau \to 0$.

It is worth while to point out a remarkable property of the slowing down density as given by (1.15): the function $Q(\boldsymbol{r}'; \boldsymbol{r}_k; \tau)$ is symmetrical with respect to the interchange of the points $\boldsymbol{r}'$ and $\boldsymbol{r}_k$. This fact is nothing but a consequence of the general reciprocity theorem of the transport theory of neutrons, holding for the age approximation as well. The probability of finding a neutron with energy $E < E_0$ in the volume element $\mathrm{d}\boldsymbol{r}'$ at $\boldsymbol{r}'$, due to the presence of a unit neutron source of energy E_0 at the point $\boldsymbol{r}_k$ has the same value as the probability of finding a neutron in $\mathrm{d}\boldsymbol{r}_k$, when the unit fast source is located at $\boldsymbol{r}'$. Looking at $\bar{Q}$ as a Green's function, the above symmetry properties could have been forseen on a strictly mathematical ground.

1·2. *Diffusion kernel.* – In absence of sources the diffusion equation in the absorbing moderator is formally similar to the $\mathscr{L}_\tau$-transform of eq. (1.1), and the diffusion kernel is required to vanish on the moderator's surface $r = R$, the same way as the function $\bar{Q}$ does.

Let $f(\boldsymbol{r}; \boldsymbol{r}')$ represent the above mentioned kernel, *i.e.* the thermal flux at $\boldsymbol{r}$ due to a unit, time independent, source of thermal neutrons located at $\boldsymbol{r}'$.

The function f has to satisfy the following diffusion equation

$$\nabla_r^2 f(\boldsymbol{r}; \boldsymbol{r}') - \varkappa^2 f(\boldsymbol{r}; \boldsymbol{r}') + \frac{1}{D} \delta(z - z') \cdot \frac{1}{r'} \cdot \delta(r - r') \cdot \delta(\theta - \theta') = 0 \,, \tag{1.18}$$

where the last term represents the source distribution and, by definition, $\varkappa^2 = 1/L^2 \equiv \Sigma_a / D$.

A comparison with eq. (1.14) suggests that we can simply take

$$(1.19)\qquad f(\boldsymbol{r};\boldsymbol{r}') = \frac{1}{2\pi R^2}\sum_{-\infty}^{+\infty}{}_n \cos n(\theta-\theta')\cdot \sum_{1}^{\infty}{}_l \frac{\exp\left[-|z-z'|\cdot\sqrt{\frac{\alpha_{nl}^2}{R^2}+\varkappa^2}\right] J_n\left(\frac{\alpha_{nl}}{R}r\right) J_n\left(\frac{\alpha_{nl}}{R}r'\right)}{D\sqrt{\frac{\alpha_{nl}^2}{R^2}+\varkappa^2}\,[J_n'(\alpha_{nl})]^2}.$$

We shall then verify that eq. (1.18) turns out to be satisfied by (1.19).

To perform this calculation properly, we remark that $f(\boldsymbol{r};\boldsymbol{r}')$ is a continuous function of z, while its first derivative has a step discontinuity in the points $z=z'$.

This fact requires the addition of a singular term of the form $\delta(z-z')$ when the second order derivative of f with respect to z is computed. We shall then obtain

$$\nabla_r^2 f(\boldsymbol{r};\boldsymbol{r}') = \frac{1}{2\pi R^2}\sum_{-\infty}^{+\infty}{}_n \cos n(\theta-\theta')\cdot \sum_{1}^{\infty}{}_l \frac{J_n\left(\frac{\alpha_{nl}}{R}r\right)\cdot J_n\left(\frac{\alpha_{nl}}{R}r'\right)\cdot\left[\varkappa^2-2\sqrt{\frac{\alpha_{nl}^2}{R^2}+\varkappa^2}\cdot\delta(z-z')\right]\cdot\exp\left[-|z-z'|\cdot\sqrt{\frac{\alpha_{nl}^2}{R^2}+\varkappa^2}\right]}{D\sqrt{\frac{\alpha_{nl}^2}{R^2}+\varkappa^2}\cdot[J_n'(\alpha_{nl})]^2}$$

so that, remembering the expansion (1.17), equation (1.18) is immediately seen to be satisfied; in effect, by definition: $\delta(z)\cdot e^{\alpha z}\equiv\delta(z)$, for all values of α.

1·3. *Thermal flux due to a fast source.* – To determine the thermal flux distribution $F(\boldsymbol{r};\boldsymbol{r}_k)$ in the point $\boldsymbol{r}$ of the cylindrical moderating region due to the presence of a unit fast neutron source in $\boldsymbol{r}_k$, one has to perform the following integration

$$(1.20)\qquad F(\boldsymbol{r};\boldsymbol{r}_k) = \iiint_{\text{mcd. vol.}} f(\boldsymbol{r};\boldsymbol{r}')\cdot Q(\boldsymbol{r}';\boldsymbol{r}_k;\tau_{\text{th}})\cdot \mathrm{d}\boldsymbol{r}'\,,$$

because a neutron can thermalize anywhere in the system, according to the probability distribution Q.

By remembering the following results:

$$(a_-)\qquad \int_0^{2\pi}\cos m(\theta_k-\theta')\cdot\cos n(\theta'-\theta)\,\mathrm{d}\theta' = \pi\cdot\cos m(\theta-\theta_k)\cdot\delta_{mn}\cdot(1+\delta_{m0})\,;$$

$$(b_-)\qquad \int_0^{R} J_m\left(\frac{\alpha_{mj}}{R}r'\right)\cdot J_m\left(\frac{\alpha_{ml}}{R}r'\right)\cdot r'\,\mathrm{d}r' = \frac{1}{2}R^2[J_m'(\alpha_{mj})]^2\cdot\delta_{jl}$$

and performing a term by term integration —which will be justified later —of the product series of the two multiple expansions (1.15) and (1.19) one easily gets:

$$F(\boldsymbol{r};\boldsymbol{r}_k)=\frac{1}{2\pi R^2}\sum_{-\infty}^{+\infty}{}_m\cos m(\theta-\theta_k)\cdot[1+\delta_{m0}]\cdot \cdot\sum_{1}^{\infty}{}_j\frac{Z_{mj}(|z-z_k|)\cdot\exp\left[-\frac{\alpha_{mj}^2}{R^2}\tau_{\mathrm{th}}\right]}{[J'_m(\alpha_{mj})]^2}J_m\left(\frac{\alpha_{mj}}{R}r\right)\cdot J_m\left(\frac{\alpha_{mj}}{R}r_k\right). \tag{1.21}$$

In the above formula the factor $Z_{mj}(|z-z_k|)$ depending just on the distance between the planes $z=$ constant, on which the source and the observation point are located, is defined as follows:

$$Z_{mj}(|z-z_k|)\equiv\int_{-\infty}^{+\infty}\frac{\exp\left[-\frac{(z_k-z')^2}{4\tau}\right]}{2\sqrt{\pi\tau}}\cdot\frac{\exp\left[-|z-z'|\sqrt{\frac{\alpha_{mj}^2}{R^2}+\varkappa^2}\right]}{2D\cdot\sqrt{\frac{\alpha_{mj}^2}{R^2}+\varkappa^2}}\cdot\mathrm{d}z'= \tag{1.22}$$

$$=\frac{\exp\left[\tau\left(\frac{\alpha_{mj}^2}{R^2}+\varkappa^2\right)\right]}{4D\cdot\sqrt{\frac{\alpha_{mj}^2}{R^2}+\varkappa^2}}\left\{\exp\left[-(z-z_k)\sqrt{\frac{\alpha_{mj}^2}{R^2}+\varkappa^2}\right]\cdot\right.$$

$$\cdot\left(1+\mathrm{Erf}\left\{\frac{1}{2\sqrt{\tau}}(z-z_k)-\sqrt{\tau}\sqrt{\frac{\alpha_{mj}^2}{R^2}+\varkappa^2}\right\}\right)+\exp\left[(z-z_k)\sqrt{\frac{\alpha_{mj}^2}{R^2}+\varkappa^2}\right]\cdot$$

$$\left.\cdot\left(1-\mathrm{Erf}\left\{\frac{1}{2\sqrt{\tau}}(z-z_k)+\sqrt{\tau}\cdot\sqrt{\frac{\alpha_{mj}^2}{R^2}+\varkappa^2}\right\}\right)\right\}.$$

The integration with respect to z' has been worked out quite elementary, and Erf (x) stands for « Error Function » of x, defined as follows:

$$\mathrm{Erf}\,(x)=\frac{2}{\sqrt{\pi}}\cdot\int_0^x\exp\left[-t^2\right]\mathrm{d}t\;;\qquad\mathrm{Erf}\,(-x)=-\mathrm{Erf}\,(x)\,.$$

1.4. *A remark on the multigroup approach.* – The diffusion kernels derived above could constitute a suitable basis for working out a multigroup-diffusion treatment of our problem.

But, to allow the calculations to be somehow similar to the above developed ones, we shall have to make only few and wide subdivisions of the energy interval; this fact is sufficient to guarantee that, for any one of the few energy groups so defined, the source neutrons are all coming from the preceding one. Further, the first energy group shall be required to include all the fission neutrons.

2. – Epithermal and thermal leakage probabilities.

2·1. *Epithermal energy region.* – Before going into the details of the heterogeneous reactor theory we want to give some additional remarks concerning the leakage and non-leakage probability during thermalization and diffusion, as a function of the source position. This question is intimately connected with the «importance function» of neutrons, a rough evaluation of which might be given, at least for the homogeneous multiplying medium, using the first order perturbation theory ([3]). The following treatment is the exact one, and applies in age-diffusion approximation as well as in multigroup diffusion theory, within the limits stated before.

Starting from eq. (1.15) let us evaluate, first of all, the fast leakage probability P_{L} for a neutron born at $\boldsymbol{r}_k$ with age $\tau = 0$, which corresponds to the energy E_0 and the lethargy $u = 0$. By definition, if $\Phi(\boldsymbol{r}'; \boldsymbol{r}_k; u)$ represents the flux per unit lethargy interval, due to a unit fast source in $\boldsymbol{r}_k$, we shall have:

$$
\begin{aligned}
(2.1)\qquad P_{\mathrm{L}} = -\int_0^{u_{\mathrm{th}}} D(u) \Big[\iint_{\mathrm{cyl.\ surf.}} \mathrm{grad}_{\boldsymbol{r}'}\, \Phi(\boldsymbol{r}'; \boldsymbol{r}_k; u) \times \boldsymbol{n} \cdot \mathrm{d}s \Big] \mathrm{d}u = \\
-\int_0^{u_{\mathrm{th}}} D(u) \Big[\iiint_{\mathrm{mod.\ vol.}} \nabla^2_{\boldsymbol{r}'}\, \Phi(\boldsymbol{r}'; \boldsymbol{r}_k; u) \cdot \mathrm{d}\boldsymbol{r}' \Big] du = \\
-\int_0^{u_{\mathrm{th}}} D(u) \Big[\iiint_{\mathrm{mod.\ vol.}} \frac{1}{\xi \Sigma_s(u)} \cdot \nabla^2_{\boldsymbol{r}'}\, Q(\boldsymbol{r}'; \boldsymbol{r}_k; u) \cdot \mathrm{d}\boldsymbol{r}' \Big] \mathrm{d}u = \\
= -\int_0^{\tau_{\mathrm{th}}} \Big[\iiint_{\mathrm{mod.\ vol.}} \frac{\partial Q(\boldsymbol{r}'; \boldsymbol{r}_k; \tau)}{\partial \tau}\, \mathrm{d}\boldsymbol{r}' \Big] \mathrm{d}\tau \,.
\end{aligned}
$$

In the above calculation the age equation (1.1) together with the following equalities have been used:

$$
(a_-)\qquad \Phi(\boldsymbol{r}; u) = \frac{1}{\xi \cdot \Sigma_s(u)}\, Q(\boldsymbol{r}; u) \,,
$$

connecting epithermal flux and slowing down density in a non absorbing moderator; ξ stands here for the average logarithmic energy decrement per collision, $\Sigma_s(u)$ is the macroscopic scattering cross-section of the moderator; the unit

([3]) A. M. Weinberg and E. P. Wigner: *The physical theory of neutron chain reactors* (Chicago, 1958).

vector $\boldsymbol{n}$ is the outgoing normal on the ciylnder's surface;

$$(b_{-}) \qquad d\tau = \frac{D(u)}{\xi \cdot \Sigma_s(u)}\, du\,,$$

together with the boundary condition $\tau(0) = 0$, defining the age τ as a function of the lethargy u.

In order to avoid the introduction of an improper function, when evaluating the integral (2.1), it will be sufficient to perform firstly the spatial integration of the derivative of Q with respect to τ, and, secondly, to integrate from ε to τ_{th} with respect to τ. The resulting value of the integral function, in the limit $\varepsilon \to 0$ is as follows:

$$(2.2) \qquad P_L(r_k) = 2 \sum_1^\infty{}_j \frac{J_0\left((\alpha_{0j}/R) r_k\right)}{J_1(\alpha_{0j}) \cdot \alpha_{0j}} - 2 \sum_1^\infty{}_j \frac{\exp\left[-\dfrac{\alpha_{0j}^2}{R^2}\tau_{th}\right]}{J_1(\alpha_{0j}) \cdot \alpha_{0j}} \cdot J_0\left(\frac{\alpha_{0j}}{R} r_k\right).$$

To evaluate the fast non-leakage probability P_{NL} we simply integrate the slowing down density at thermal age over the cylindrical moderating volume. Using formula (1.15) again, we deduce that:

$$(2.3) \qquad P_{NL}(r_k) = \iiint_{\text{mod. vol.}} Q(\boldsymbol{r}'\,;\, \boldsymbol{r}_k\,;\, \tau_{th})\, d\boldsymbol{r}' = 2 \sum_1^\infty{}_j \frac{\exp\left[-\dfrac{\alpha_{0j}^2}{R^2}\tau_{th}\right]}{J_1(\alpha_{0j}) \cdot \alpha_{0j}} \cdot J_0\left(\frac{\alpha_{0j}}{R} r_k\right).$$

As the moderator has been assumed to be a non absorbing one for epithermal neutrons, the sum of P_L and P_{NL} is expected to take the unit value. On the other hand, by adding together (2.2) and (2.3), one easily gets:

$$(2.4) \qquad P_L(r_k) + P_{NL}(r_k) = 2 \sum_1^\infty{}_j \frac{J_0\left((\alpha_{0j}/R) r_k\right)}{J_1(\alpha_{0j}) \cdot \alpha_{0j}}\,.$$

Nevertheless the condition to be satisfied by the sum of the two probabilities is completely fulfilled, because the right hand side of eq. (2.4) represents nothing but the Hankel series expansion of the constant function $\mathbf{1}(r_k)$.

Its value is thus independent of r_k and the difference from 1 of any partial sum of the series (2.4) can be made as small as we want, by increasing the number of the terms taken into account.

A general remark concering leakage and non-leakage probabilities in the present geometry is suggested by the expressions (2.2) and (2.3): when evaluating the total fast neutron current leaking out of the system, only the zero

order angular harmonic gives a significant contribution; the same happens, obviously, for the probability of thermalization inside the cylinder, as well as for thermal absorption or leakage probabilities given below.

2·2. *Thermal energy region.* – The thermal leakage probability p_L for a neutron thermalized in $\boldsymbol{r}'$ is easily evaluated starting from formula (1.19):

$$(2.5)\qquad p_L(r') = -D \iint_{\text{cyl. sur.}} \operatorname{grad}_{\boldsymbol{r}} f(\boldsymbol{r};\boldsymbol{r}') \times \boldsymbol{n}\cdot \mathrm{d}S =$$

$$= -2D \int_0^\infty \mathrm{d}(z-z')\cdot \int_0^{2\pi} \left[\frac{\partial}{\partial r} f(\boldsymbol{r};\boldsymbol{r}')\right]_{r=R} \cdot R\,\mathrm{d}\theta = 2\sum_1^\infty{}_l \frac{J_0((\alpha_{0l}/R)r')}{J_1(\alpha_{0l})\cdot\alpha_{0l}} -$$

$$- 2\varkappa^2 \sum_1^\infty{}_l \frac{J_0((\alpha_{0l}/R)r')}{[(\alpha_{0l}^2/R^2)+\varkappa^2]\cdot J_1(\alpha_{0l})\cdot\alpha_{0l}} .$$

In the moderator the macroscopic absorption cross-section for thermal neutrons is Σ_a, so that the thermal absorption probability p_A comes out to be

$$(2.6)\qquad p_A(r') = 2\int_0^\infty \mathrm{d}(z-z')\cdot \int_0^{2\pi}\int_0^R f(\boldsymbol{r};\boldsymbol{r}')\,\Sigma_a\cdot r\,\mathrm{d}r\,\mathrm{d}\theta =$$

$$= 2\varkappa^2\cdot \sum_1^\infty{}_l \frac{J_0((\alpha_{0l}/R)r')}{[(\alpha_{0l}^2/R^2)+\varkappa^2]\cdot J_1(\alpha_{0l})\cdot\alpha_{0l}} .$$

The obvious property of the sum of the two probabilities still easily follows from (2.5), (2.6) and the previous remarks about the Hankel series expansion of a constant:

$$p_L(r') + p_A(r') = \mathbf{1}(r') \equiv 1 .$$

2·3. *Total leakage probability.* – Let's define now the « total leakage probability » P_L^{tot} for a neutron born at $\boldsymbol{r}_k$ as the sum of $P_L(r_k)$, the leakage probability during thermalization, plus the probability of leaking out when thermal diffusion in the moderator occurs:

$$(2.7)\qquad P_L^{tot}(r_k) = P_L(r_k) + \int_0^{2\pi}\int_0^R \left[2\cdot\int_0^\infty \mathrm{d}(z'-z_k)\cdot Q(\boldsymbol{r}';\boldsymbol{r}_k;{}^{\cdot}{}_{th}\right] \cdot p_L(r')\cdot r'\cdot \mathrm{d}r'\cdot \mathrm{d}\theta' .$$

The present formula accounts for the fact that, in our geometry, the leakage is unaffected by the value of the axial coordinate.

To evaluate $P_{\mathrm{L}}^{\mathrm{tot}}$ we use the formulas (1.15) and (2.5), together with (2.2). It easily follows

$$P_{\mathrm{L}}^{\mathrm{tot}}(r_k) = P_{\mathrm{L}}(r_k) + \frac{2}{R^2}\sum_1^\infty{}_j \frac{\alpha_{0j}^2 \cdot \exp\left[-\frac{\alpha_{0j}^2}{R^2}\tau_{\mathrm{th}}\right]\cdot J_0((\alpha_{0j}/R)r_k)}{((\alpha_{0j}^2/R^2)+\varkappa^2)\cdot J_1(\alpha_{0j})\cdot\alpha_{0j}} \quad . \tag{2.8}$$

A « total absorption probability » $P_{\mathrm{L}}^{\mathrm{tot}}$ for a neutron born at $\boldsymbol{r}_k$ can be consistently defined making use of the thermal flux distribution $F(\boldsymbol{r};\boldsymbol{r}_k)$ due to a unit fast source at $\boldsymbol{r}_k$ as given by eq. (1.21):

$$P_{\Lambda}^{\mathrm{tot}}(r_k) = 2\cdot\int_0^\infty \mathrm{d}(z-z_k)\int_0^{2\pi}\int_0^R \Sigma_a \cdot F(\boldsymbol{r};\boldsymbol{r}_k) r\cdot\mathrm{d}r\cdot\mathrm{d}\theta = \tag{2.9}$$

$$= 2\varkappa^2 \sum_1^\infty{}_j \frac{\exp\left[-\frac{\alpha_{0j}^2}{R^2}\tau_{\mathrm{th}}\right]\cdot J_0((\alpha_{0j}/R)r_k)}{((\alpha_{0j}^2/R^2)+\varkappa^2)\cdot J_1(\alpha_{0j})\cdot\alpha_{0j}} \quad .$$

By summing up eq. (2.8) and (2.9), taking into account (2.2), we still have:

$$P_{\mathrm{L}}^{\mathrm{tot}}(r_k) + P_{\Lambda}^{\mathrm{tot}}(r_k) = \mathbf{1}(r_k) \equiv 1 \; . \tag{2.10}$$

This fact, as formulas (2.2) and (2.8) have been fully justified in the preceding development, can be taken as an indirect proof of the correctness of eq. (1.21). On the other hand a detailed proof of the arguments, from which (1.21) has been derived, would have involved a much more cumbersome work.

3. – The heterogeneous treatment.

The thermalization and diffusion kernels being known, we can apply now the principles of the heterogeneous theory (4).

Let us consider a set of well-separated, small fuel lumps, embedded in the moderating cylinder.

Each one of them can be imagined to be enclosed into an highly symmetric, hull surface—let's say, a spherical or cylindrical one—in which no appreciable portion of the elastically moderating material is present. A thermal neutron entering such a hull surface, either can be returned to the moderator without any energy change, or can be absorbed inside the surface itself.

(4) S. M. Feinberg: *Heterogeneous methods for calculating reactors*, in *Proc. of the Intern. Conference on the Peaceful Uses of Atomic Energy*, Geneva, 1955, P/669, vol. 5 (New York, 1956), p. 484.

In this case, it may happen that, due to the fission cross section, which is assumed to be present in the inside volume, one or more neutrons are returned to the moderator, at higher energy level.

Neglecting for the moment the inelastic collisions in the fuel, we assume that the fast neutrons start to thermalize as soon as they enter the moderator: during thermal diffusion they can be absorbed in any multiplying or absorbing lump of the system, as well as in the moderator itself, if no leakage did occur, during thermalization and diffusion.

We shall call simply *block* any multiplying or purely absorbing lump, surrounded by its fictitious hull surface.

Some characteristic parameters concerning a block will be defined below.

Let's call σ_k the hull surface enclosing the k-th lump. As the equations of neutron diffusion are linear, we assume the following equality to hold:

$$\hat{\Phi}(\sigma_k) = \gamma_k \cdot I_k \,, \tag{3.1}$$

where: $\hat{\Phi}(\sigma_k)$ represents the space-averaged thermal flux on σ_k;

I_k is the number of thermal neutrons absorbed by the k-th block in the unit time, under stationary conditions;

γ_k is a parameter characterizing the k-th block. Its inverse provides a measure of the probability of absorption in the k-th block experienced by a thermal neutron diffusing on the surface σ_k. This parameter is essentially depending on the fuel and absorber's concentration into the block volume, and, if diffusion approximation is justified, it can be related to the *extrapolation length* of the block in a straitforward way.

Let I_k thermal neutrons be abosrbed in the K-th block: as a consequence q_k fast neutrons are released: to define the multiplying properties of the block we use a parameter η_k given by the following equation:

$$q_k = \eta_k \cdot I_k \,. \tag{3.2}$$

Its value accounts for the fuel properties—mainly the enrichment—and for the presence of structural materials in the fuel lump also. It obviously vanishes when the *block* is a *purely absorbing* one. In a multiblock system both sets of parameters η's and γ's shall have to be given by experiments, or detailed neutron transport calculations. *Any subset of them can be chosen as eigenvalues of the problem.*

The aim of the *heterogeneous theory* is to evaluate the average thermal flux on the contour of any block, in order to state a *criticality condition*, connecting

the nuclear properties and the mutual positions of the blocks themselves, together with the moderator's characteristics and dimensions. When the proper choice of the eigenvalues has been made through the critical equation, it comes out as a consequence that a stationary flux distribution, is allowed in the system, the excess neutrons produced by the block interaction being exactly balanced by the fast and thermal leakages, and by the moderator's capture.

3·1. *Small spherical blocks.* – Let the linear dimensions of the neutron source-sinks we are dealing with be small, with respect to the neutron diffusion lenght in the moderator. In this case the fuel and absorber's lumps may be enclosed into some small spheres, which will be considered in the following as *point singularities* of the neutron flux distribution, unless the contrary is explicitly stated.

Under stationary conditions we know from eq. (1.21) how to evaluate the thermal flux in the moderator due to a fast source. But the blocks are acting also as lumped absorbers of thermal neutrons. We describe the thermal neutron depletion, caused in the moderator by this localized absorption, by considering any block as a *negative source* of thermal neutrons, having the same intensity as the effective sink.

Let M blocks be present in the system; if no resonance capture occurs, by remembering formulas (1.19) and (1.21), the effective thermal flux Φ at the space point $\boldsymbol{r}$ is easily seen to be given by

$$\Phi(\boldsymbol{r}) = \sum_{1}^{M}{}_k \left[g_k F(\boldsymbol{r}; \boldsymbol{r}_k) - I_k \cdot f(\boldsymbol{r}; \boldsymbol{r}_k) \right] . \tag{3.3}$$

Now, if $\boldsymbol{r}$ approaches a point on the surface σ_h, taking into account (3.1) and (3.2), we shall have

$$\gamma_h \cdot I_h = \sum_{1}^{M}{}_k I_k \cdot F_{hk} \cdot \eta_k - f_{hk}] \equiv \sum_{1}^{M}{}_k d_{hk} \cdot I_k , \tag{3.4}$$

with the following meaning for the symbols:

$$d_{hk} = \frac{1}{\pi R^2} \sum_{-\infty}^{+\infty}{}_m \cos m(\theta_h - \theta_k) \cdot \sum_{1}^{\infty}{}_j \frac{J_m\left((\alpha_{mj}/R) r_h\right) \cdot J_m\left((\alpha_{mj}/R) r_k\right)}{[J'_m(\alpha_{mj})]^2} \cdot \tag{3.5}$$

$$\cdot \left\{ \frac{1 + \delta_{m0}}{2} \cdot \eta_k \cdot \exp\left[- \frac{\alpha_{mj}}{R^2} (\tau_{th} - \tau_k^0) \right] \cdot Z_{mj}(|z_h - z_k|; \tau_{th} - \tau_k^0) - \frac{\exp\left[-|z_h - z_k| \sqrt{\frac{\alpha_{mj}^2}{R^2} + \varkappa^2} \right]}{2D \sqrt{\frac{\alpha_{mj}^2}{R^2} + \varkappa^2}} \right\}$$

τ_k^0: The energy at which the fast neutrons are released by the k-th block is is assumed to be $E(\tau_k^0)$. So our theory applies easily even when different kinds of fuel, with different fission spectra or inelastic cross-section, are present in the system.

In equation (3.4) the assumption is made that the average flux on σ_h due to any block other than the h-th may be accurately represented by its value in $\boldsymbol{r}_h$, the central point of σ_h.

But, when considering the action of a certain block on itself, in order to avoid the appearance of singular functions, we shall evaluate in detail the average flux on the contour, at least for its negative component f. So, if $|\boldsymbol{\rho}_h|$ represents the radius of the h-th block, we define f_{hh} according to the following equation:

$$f_{hh} \quad \frac{1}{4\pi\varrho_h^2}\iint_{\sigma_h} f(\boldsymbol{r}_h + \boldsymbol{\rho}_h; \boldsymbol{r}_h)\, \mathrm{d}\sigma_h = \frac{1}{2\pi R^2}\sum_{-\infty}^{+\infty m}\sum_{1}^{\infty j}\frac{J_m^2((\alpha_{mj}/R)\, r_h)\cdot I_{mj}(\varrho_h)}{D\sqrt{\dfrac{\alpha_{mj}^2}{R^2}+\varkappa^2}\,[J'_m(\alpha_{mj})]^2}\,.$$

Here the I_{mj}'s stand for the definite integrals

$$I_{mj}(\varrho_h) = \int_0^{\pi/2}\exp\left[-\varrho_h\cdot\cos\varphi\sqrt{\frac{\alpha_{mj}^2}{R^2}+\varkappa^2}\right]\cdot J_0\left(\frac{\alpha_{mj}}{R}\,\varrho_h\,\sin\varphi\right)\cdot\sin\varphi\cdot\mathrm{d}\varphi$$

a previous evaluation of which is required.

To allow the homogeneous system of linear equations (3.4) to be consistent, the determinant of the coefficients is required to vanish:

$$\|F_{hk}\cdot\eta_k - f_{hk} - \gamma_h\cdot\delta_{hk}\| = 0 \qquad (3.6)$$

This condition is *the critical equation* for our system, and can be used to determine the eigenvalues. Then, going back to (3.4), one can evaluate the intensity of thermal absorption I_k in any block.

In the very particular case in which i) all the blocks have the same nuclear properties, ii) the parameter η has been given once for all, and iii) γ is chosen as eigenvalue, the resolution of the equations (3.4) reduces to the standard eigenvalues problem for the matrix $\{d\}$, whose elements are the d_{hk} given by (3.5). It goes without saying that, in this case, the eigenvalue γ to be considered is the real and positive root of the characteristic equation which corresponds to a set of solutions I_k all real and positive.

In connection with problems of this type we want to point out explicitly that the nuclear parameters are not the only ones that can be taken as eigenvalues: the mutual distances $\boldsymbol{r}_{hk}$ of the blocks, or the radius R, for instance could play the same role in the theory.

3·2. *Linear blocks.* – Let us consider now a multiplying structure consisting of a certain number of fuel elements of the type of *rods*, *hollow cylinders or clusters*, which we shall assume for the present to be axially infinite, parallel to each other and to the axis of the moderating system. If the radii of all of those fuel elements are small, as compared with their mutual distances, and with the neutron mean free path in the moderator, it will be justified to consider any fuel lump as a « *line source-sink singularity* » of the neutron flux distribution. The parameters η's and γ's, defined before, and the intensity of thermal neutron absorption I_k shall have to be referred to the unit length of a block.

The analysis of the block configuration we are now dealing with could have been afforded in plane geometry. Nevertheless we shall still use the general kernels evaluated in Part I, after a suitable integration with respect to the (axial coordinate.

The same considerations, made for the small spherical blocks, lead us to the following system of homogeneous linear equations

$$\gamma_h \cdot I_h = \sum_1^M{}_k I_k [F^*_{hk} \cdot \eta_k - f^*_{hk}] = \sum_1^M{}_k d^*_{hk} \cdot I_k \,. \tag{3.7}$$

Now the d^*'s are given by

$$d_{hk} = \frac{1}{\pi R^2} \sum_{-\infty}^{+\infty}{}_m \cos m(\theta_h - \theta_k) \cdot \cdot \sum_1^\infty{}_j \frac{J_m\left((\alpha_{mj}/R)\, r_h\right) \cdot J_m\left((\alpha_{mj}/R)\, r_k\right)}{D\left((\alpha^2_{mj}/R^2) + \varkappa^2\right) [J'_m(\alpha_{mj})]^2} \left\{ \eta_k \frac{1 + \delta_{m0}}{1} \exp\left[- \frac{\alpha^2_{mj}}{R^2} (\tau_{\mathrm{th}} - \tau^0_k) \right] - 1 \right\}, \tag{3.8}$$

for $h \neq k$, and for $h = k$, the kernel f_{hh} being averaged on the cylindrical surface σ_h of radius ϱ_h, the term $[-1]$ in the last factor of (3.8), has to be substituted by $[-J_0\left((\alpha_{0j}/R)\varrho_h\right) \cdot \delta_{m0}]$.

In order to reach an even finer degree of precision, accounting —at least approximately—for the finite dimensions of the blocks, one can evaluate in detail the average value of the flux—coming from any point source at $\boldsymbol{r}_k$—on the contour of the h-th block. This operation simply causes the general term of the series (3.8) to be multiplied by a factor $J_0\left((\alpha_{mj}/R)\varrho_h\right)$.

The heterogeneous theory can be worked out also for multiplying structures of finite height. An axial cosine distribution of the macroscopic flux can be assumed in all the reactors in which the linear blocks give rise to a quite re-

gular and thick lattice on the core cross-section. But the axial finiteness can be accounted for exactly, if required. This can be done by the *image source method*, which implies, in our geometry, the substitution of the following formula to the (1.4):

$$(3.9)\qquad Q_{\text{sing}}(\boldsymbol{r}'; \boldsymbol{r}_k; \tau) = \frac{\exp[-\varrho^2/4\tau]}{(\sqrt{4\pi\tau})^3} \cdot \sum_{-\infty}^{\infty}{}_{\nu} \left[\exp\left[-\frac{(z' - z_k - 4\nu L)^2}{4\tau} \right] - \exp\left[-\frac{\{z' + z_k - 4(\nu + \frac{1}{2})L\}^2}{4\tau} \right] \right].$$

Here the planes $z' = \pm L$ are the axial moderator's boundaries. Obviously all the formulas depending on (1.4) shall have to be modified according to (3.9).

We want to remark explicitly that *the order of the critical determinants* of the type (3.6) is by no means coincident with the number of blocks constituting the multiplying system. It happens in almost all the practical cases that the blocks can be subdivided in *classes*, according to their nuclear properties and geometrical positions. Due to the complete equivalence of all the blocks belonging to the same class, it will be sufficient to apply the boundary conditions of the type (3.4) on the contour of one block for each class. This considerable simplification allows the heterogeneous techniques to be applied, even when their use could seem unpractical.

3·3. *The extrema of functions of the critical mass.* – A problem of great physical importance, both from the theoretical and practical viewpoint, is that of finding the maxima or minima of given functions of the critical mass. Let now the blocks of our system be distributed in M classes, the k-th one of them containing N_k elements.

Any functions of the critical mass (CM)—for instance, the critical mass itself, its total weight or total cost—shall be a function of the following type:

$$(3.10)\qquad G(\text{CM}) = G(N_1 \cdot \eta_1, N_2 \cdot \eta_2, \ldots, N_M \cdot \eta_M; N_1 \cdot \gamma_1, N_2 \cdot \gamma_2, \ldots, N_M \cdot \gamma_M).$$

In a fuel element the parameters η and γ are depending on each other, and we assume the following M functions φ_h, for instance, to be known:

$$(3.11)\qquad \frac{\partial \gamma_h}{\partial \eta_h} = \varphi_h(\eta_h; \delta_1, \delta_2, \ldots, \delta_n) \qquad h = 1, 2, \ldots M.$$

Here $\delta_1, \delta_2, \ldots, \delta_n$ represent an additional set of n independent variables —let's say, concentrations of absorbers—on which the γ's may still be dependent.

Furthermore, taking into account the final remarks of the preceding paragraph, concerning the order of the critical determinants in a multi-class heterogeneous structure, we want to write down explicitly the standard type of critical equation.

We label each block by two suffixes—let's say « k » and « u »—the first one of them indicating which class the block is belonging to, and the second one giving the position of the block into its own class. We make now the following convention: the boundary conditions are to be imposed on the contour of the first block of any class.

So, the analogous of the system of equations (3.4), under the same restrictions specified there, will become, for a multi-class structure of blocks:

$$\gamma_h \cdot I_h = \sum_1^M{}_k I_k \cdot \left[\eta_k \cdot \sum_1^{N_k}{}_u F_{h1;ku} - \sum_1^{N_k}{}_u f_{h1;ku} \right] . \tag{3.12}$$

With the following definitions

$$\begin{cases} \sum_1^{N_k}{}_u F_{h1;ku} - S_{h;k} \\ \sum_1^{N_k}{}_u f_{h1;ku} - s_{h;k} \end{cases} \tag{3.13}$$

the standard form of the critical equation still comes out to be

$$\Delta = \| S_{h;k} \cdot \eta_k - s_{h;k} - \gamma_h \cdot \delta_{hk} \| = 0 . \tag{3.14}$$

This equation is obviously not dependent on the above convention of imposing the boundary condition around the « first block » of any class; but the remarkable property of symmetry of the critical determinant with respect to its main diagonal is not present, even if the γ_i's are coincident with each other.

We want now to find the maxima and minima of G(CM) for a system in which the geometrical parameters, as well as the moderator's characteristics are given, and all the η's—or any subset of them—have to be considered as eigenvalues. The only restrictions on the η_i's are the following ones, coming from physical considerations: $a_h < \eta_h < b_h$, the a's and b's being some given constants.

We suppose also that a set of « initial values » for the γ's has been assigned before, in connection with a certain arbitrary choice of the η_h's — η_h^0's. But, as the η_h^0's are not likely to be the eigenvalues of the problem, the γ's also will vary, so as to take in the critical system a set of values consistent with the equations (3.11).

The stationary points of (3.10) as a function of the η's, subjected to the restriction (3.14) are easily found by means of the *Lagrange's method of multipliers*.

Let P of the M η's be choosen as eigenvalues: the function $H(\eta_1, \eta_2, ..., \eta_P; \lambda) \equiv G(\mathrm{CM}) + \lambda \Delta$, depending on the $P+1$ variables η_1, η_2, ..., η_P; λ, (λ is the multiplier), is stationary in the points satisfying the following $P+1$ equations:

$$(3.15) \qquad \begin{cases} \dfrac{\partial H}{\partial \eta_l} = \left[\dfrac{\partial G(\mathrm{CM})}{\partial \eta_l}\right]^* + \lambda \cdot \sum_h^M A_{hl} \cdot \left[S_{h;l} - \dfrac{\partial \gamma_h}{\partial \eta_l} \cdot \delta_{hl}\right] = 0\,; \quad l = 1,\ 2,\ ...,\ P. \\ \dfrac{\partial H}{\partial \lambda} = \Delta = 0\,, \end{cases}$$

the A_{hl} being the cofactors of the (hl)-th elements of the critical determinant, and $[\partial G/\partial \eta_l]^* \equiv \partial G/\partial \eta_l + (\partial G/\partial \gamma_l) \cdot (\partial \gamma_l/\partial \eta_l)$.

The equations (3.15) constitute a set of *necessary* conditions for the presence of an extremum of $G(\mathrm{CM})$. A further discussion is required in order to recognize if this point is a maximum or a minimum. But this can now be done following the classical lines of the theory of functions of several variables.

RIASSUNTO

Questo lavoro contiene la soluzione teorica dei problemi di criticità e di distribuzione stazionaria del flusso neutronico termico per una struttura moltiplicante eterogenea di forma cilindrica, nella quale i blocchi di materiale fissile siano distribuiti in modo arbitrario. L'equazione critica è ricavata col metodo dei *pozzi-sorgente*, trattando la termalizzazione dei neutroni con la teoria dell'età oppure con la diffusione a più gruppi. Vengono calcolate le probabilità di fuga e di assorbimento per neutroni nati in qualsiasi punto della struttura. È descritto un metodo per la ricerca *simultanea* della criticità e dei massimi e minimi di qualsiasi funzione (ad esempio, peso o costo totali) della massa critica.

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A Measurement of the Total Absorption Rate of Muons in Carbon (*).

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Summary. — Negative muons stopped in a propane bubble chamber form mesic atoms with carbon nuclei. Subsequently they either decay by their usual mode or interact with the nucleus. Based on a sample of 2 519 mesons, the probability for interaction is found to be $(7.4 \pm 0.8)\%$, and the total absorption rate of muons in carbon becomes $(0.36 \pm 0.04) \cdot 10^5\ s^{-1}$. The result is found to be in satisfactory agreement with theory.

1. – Introduction.

When a negative muon is brought to rest in matter, it is captured into Bohr orbits about a nucleus. Later it either decays through its usual mode into an electron and two neutrinos, or alternatively interacts. The rate at which the latter process can proceed is a function of the charge Z, of the nucleus ([1,2]).

Several authors have measured the interaction rates of muons with different nuclei ([3-5]). The usual method has been to stop the mesons in a target

(*) This work was performed under the auspices of the U. S. Atomic Energy Commission.

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of the required material, and then detect the emergent decay electrons with counters. From the number of muons entering the target and the number of decay electrons leaving, the proportion of muons interacting could be estimated. Such measurements have yielded accurate values for most nuclei investigated.

However, with light elements, the decay process is dominant and difficulties arising from background effects are encountered. For carbon, only about 10% of the mesons interact, and the capture rate has not been determined to much better than $\pm 25\%$. This is particularly unfortunate in that a great deal of interest has recently centered upon muon capture in carbon. The reaction which leads to the ground state of ^{12}B provides a test of the universal Fermi interaction theory ([6]), and recent measurements of the partial rate for this process have depended to some extent upon the rather poorly known total absorption rate ([7]). The determination of this latter quantity can however be improved by using a propane bubble chamber.

A negative muon, stopped in propane (C_3H_8), is captured into orbits about a carbon nucleus ([8]). The presence or absence of an electron at the end of the meson's track gives a direct indication as to whether the muon decayed or interacted after stopping in what is effectively a carbon target. Counting the number of mesons that decay, N_d, and the number that interact, N_i, and assuming a value for the decay lifetime, τ_d, one can obtain the total absorption rate from the expression

$$\Lambda_i = \frac{N_i}{N_d \cdot \tau_d} . \tag{1}$$

2. – Method.

The photographs examined were taken with the 30 in. propane chamber operating in a magnetic field of 13 kG. No special exposure was made for this experiment; instead, film already existing of the 1.15 GeV/c K^--meson beam was used. The K-mesons have a background consisting mainly of muons, some of which stop in the chamber and are suitable for our purpose. As can

([6]) A. Fujii and H. Primakoff: *Nuovo Cimento*, **12**, 327 (1959); L. Wolfenstein: *Nuovo Cimento*, **13**, 319 (1959).

([7]) T. N. K. Godfrey: *Phys. Rev.*, **92**, 512 (1953); W. Love, S. Marder, I. Nadelhaft, R. Siegel and A. E. Taylor: *Bull. Am. Phys. Soc.*, **4**, 81 (1959); J. G. Fetkovich, T. H. Fields and R. L. McIlwain: *Bull. Am. Phys. Soc.*, **4**, 81 (1959); J. O. Burgman, J. Fischer, B. Leontic, A. Lundby, R. Meunier, J. P. Stroot and J. D. Teja: *Phys. Rev. Lett.*, **1**, 469 (1958); and H. V. Argo, F. B. Harrison H. W. Kruse and A. D. McGuire: *Phys. Rev.*, **114**, 626 (1959).

([8]) W. K. H. Panofsky, L. Aamodt and H. F. York: *Phys. Rev.*, **78**, 825 (1950).

be inferred from the results presented later, this background is composed of approximately 90% muons and 10% pions.

While only comparatively few of the muons interact, all the stopped pions, of course, are captured. This makes it imperative that one should be able to correct adequately for the latter.

As a first step, the proportion of pions accepted into the sample is reduced by using the magnetic curvature of the tracks. In order to ensure a sufficient length of track for measurement, only particles having a visible range of greater than 25 cm in the chamber are considered. A template was prepared which shows the expected magnetic curvature of a muon track over this final length. Because of multiple scattering, about half the muons in practice have a curvature greater than this average value, the remainder having less. It is because of multiple scattering that some pion tracks simulate those of muons. A rigid criterion is adopted whereby only those tracks are accepted that have a greater curvature than that indicated by the template. This procedure, while rejecting many legitimate muons, makes pion acceptance improbable. The actual extent of the reduction in the pion contamination was determined in a subsidiary experiment. This consisted of taking tracks of positive pions, readily identifiable by their characteristic decay mode, and comparing them with a template for a positive muon. It was found that only $(7 \pm 2)\%$ of the pions had a greater curvature than that expected for the muon. As half the muons satisfy the criterion, the method of selection reduces the background by a factor of seven. Of the particles accepted into the sample, therefore, a little over 1% are pions.

The correction for these remaining pions is based upon the difference in the prong distribution of the capture stars. In contrast to pions, muons only comparatively rarely produce visible prongs. MORINAGA and FRY have investigated the characteristics of stars arising from muon capture in the light elements of nuclear emulsion ([9]). It was found that about 90% of the events had no visible secondaries. This figure, however, cannot be taken as applying to our work because of the greater efficiency for detecting short tracks in nuclear emulsion. The appropriate number of zero-pronged events is, consequently, expected to be higher. Nevertheless, the results of MORINAGA and FRY are helpful. The emitted protons were found to have an energy spectrum which extended no higher than 15 MeV. In addition, the energies of the secondaries from the two or more pronged stars, were such that only rarely would more than a single prong have been resolvable had the stars occurred in a bubble chamber. It follows then that if stars are found emitting more than one prong or alternatively a single prong of energy greater than 15 MeV in

([9]) H. MORINAGA and W. F. FRY: *Nuovo Cimento*, **10**, 308 (1953).

our sample of supposed muons, these events can be discarded as examples of pion capture. However, not all pion stars are so readily recognized, because some give rise to low visible-energy releases.

In order to be able to estimate, from the observed energetic disintegrations, the number of small stars that are also present, a knowledge of the prong distributions of pion stars is required. In this connection, a subsidiary investigation was carried out on 141 pion-capture stars. The primaries for these events were identified through their having been created in Λ^0-hyperon decays. The results are shown in the first line of Table I, the single-pronged events being divided according to the energy of the prong, assuming it to be a proton. The final three columns provide a total of 62 events which have star characteristics that cannot be simulated by muon captures, and these compare with 79 events giving low visible-energy releases. The number of pions that give small stars in our sample of supposed muons will then simply be the number of mesons found to give energetic disintegrations, multiplied by the factor 1.3 ± 0.2 (*i.e.* 79/62).

TABLE I. – *Characteristics of capture stars.*

Primary particle	Number of prongs				
	0	1 [a]	1 [b]	2	3
Pions	58	21	45	14	3
Muons [c]	190	3	9	4	0

[a] Energy of the prong less than 15 MeV.
[b] Energy of the prong greater than 15 MeV.
[c] Uncorrected for pion contamination.

A confirmatory check on this estimate is provided by the curvature selection already discussed. It has been shown that 7% of all pions satisfy the criterion. Thus, if a record is made of those pions rejected, one can infer from this the number that were accepted.

Summarizing the procedure, then, tracks of length greater than 25 cm are selected if they have a curvature greater than that which a muon would have in the absence of multiple scattering. It is observed whether the particle gives a decay electron or not, and the star characteristics of any interactions are noted. An additional record is made of those particles that produce stars but do not satisfy the curvature condition.

3. – Results.

A total of 2544 stopped particles were accepted by the criterion, and, of these 2338 decayed and 206 interacted. The prong distribution of the capture stars is given in the second line of Table I.

In addition to these events, there were 129 pions that gave energetic stars but were rejected by the muon-curvature criterion. On the basis of 7% of the pions satisfying the criterion, one would have expected about ten energetic stars within the sample. It is noted from the table that there are in fact 13 events in the final three columns. Following the arguments of the previous section, this number of energetic stars indicates a total pion contamination within the sample of 30 ± 5 events [*i.e.* $(1.3 \pm 0.2)(13 \pm 3.6) + 13$]. Subtracting them from the total, the number of interactions remaining is 176, and these are attributed to muon captures.

There are several other corrections to be considered:

a) It is to be expected that there might be observational biases associated with events close to the edges of the chamber. If a decay were to occur very close to a wall or window, the electron could conceivably leave the chamber undetected. The event would then be erroneously classed as a zero-pronged interaction. On the other hand, in a similar location a particle might be mistakenly thought to have left the chamber without stopping. If so, it would have been overlooked during scanning. A study of the distribution of events throughout the chamber reveals that it is the latter effect that is important. Approximately five stars have apparently escaped detection in the regions of the chamber less than 0.5 cm from the upper and lower windows. Therefore, this number must be added to the total.

b) There are several stray Compton electrons on each picture, and if any were to accidentally coincide with zero-pronged events, the muons would appear to have decayed. This effect was found to be negligible.

c) Because of the continuous range of energies available to the electron from muon decay, some electrons might be of too small a range to be detected. A study of the expected energy spectrum shows this correction also to be insignificant.

d) A process that leads to mistaking some muon captures for decays arises when the residual nucleus undergoes β-transition following the absorption of a muon. It is expected that about 18% of muon interactions produce boron-12 through the process ([6]):

$$\mu^- + {}^{12}\mathrm{C} \to {}^{12}\mathrm{B} + \nu \,. \tag{2}$$

The ^{12}B nucleus then β-decays with a lifetime of (33.15 ± 0.2) ms according to the reaction ([10]),

$$^{12}B \rightarrow e^- + {}^{12}C + \bar{\nu} . \tag{3}$$

If this reaction happened to occur quickly, the electron's track would be photographed. It would then be impossible to distinguish the event from a direct decay of a muon. In order to be able to estimate the correction for this effect, one needs to know the time during which the β-decay can occur and still produce an electron that will leave a visible track. The interval between the beam pulse and the light flash varies between 6 and 8 ms. This separation, however, only provides an upper limit to the time available, because an electron produced near the end of this delay will not have had sufficient time in which to form a track capable of being photographed. As it is difficult to devise a precise correction, we simply assume that all decays occurring in the first 4 ms are photographed. An adequate systematic error is then included in the final result to cover both the extreme cases of either none or alternatively all of the β-decays being recorded up to 8 ms. From this effect four events should be transferred from muon decays to muon interactions.

e) Finally, it must be remembered that if impurities consisting of heavy nuclei are present in the propane, they will seriously affect the measured capture rate. There was no reason to suspect that the chemically pure propane was contamined in this way but, nevertheless, a sample of the liquid was subjected to mass spectrographic analysis at the completion of chamber operation. No impurities could be found; the analysis showed an upper limit of one in 10000 for the number of nuclei of heavy elements present compared to those of carbon.

The finally corrected data yield a total of 2519 stopped muons, of which 185 interacted and 2334 decayed.

In order to derive the capture rate, a value for the decay lifetime of the muon has to be assumed. Based on the work of several authors ([11]), the value chosen is $2.22 \cdot 10^{-6}$ s. There are several reasons why the lifetime of a bound muon should be different from that of the free meson ([12]). These effects have

([10]) E. Feenberg: *Shell Theory of the Nucleus* (Princeton, N. J., 1955), chap. 3; J. F. Vedder: *High energy beta decay of light elements*, UCRL-8324 (June 1958), p. 57.

([11]) J. Fischer, B. Leontic, A. Lundby, R. Meunier and J. P. Stroot: *Phys. Rev. Lett.*, **3**, 349 (1959); R. A. Swanson, R. A. Lundy, V. L. Telegdi and D. D. Yovanovitch: *Phys. Rev. Lett.*, **2**, 430 (1959); and W. E. Bell and E. P. Hincks: *Phys. Rev.*, **84**, 1243 (1951).

([12]) C. E. Porter and H. Primakoff: *Phys. Rev.*, **83**, 849 (1951); T. Muto M. Tanifuji, K. Inoue and T. Inoue: *Progr. Theor. Phys.*, **8**, 13 (1952); and L. Tenaglia: *Nuovo Cimento*, **13**, 284 (1959).

been shown, however, to be negligible in the case of muons bound to carbon nuclei ([13]).

Using relation (1), one finds the total absorption rate for muons in carbon to be

$$\Lambda_i = (0.36 \pm 0.04) \cdot 10^5 \text{ s}^{-1} . \tag{4}$$

The error arises in the following manner. From the number of muon interactions there is a statistical uncertainty of 7.5%. The errors in the correction for the pion contamination give a contribution amounting to 3%, while observational biases introduce a further 1%. These effects are statistically independent and yield a total of 8%. In addition there is a systematic error of about 2% associated with the correction for the β-decay of ^{12}B, and an uncertainty of 1% in the muon lifetime. The combination of all these effects yields the quoted error of 11%.

4. – Discussion.

The total absorption rate as measured in this experiment is somewhat lower than previous estimates. Using a propane chamber, Fields, McIlwain, and Fetkovich obtained a value of $0.45 \cdot 10^5$ s^{-1} based on 1 000 stopped muons ([14]). The counter technique employed by Sens ([3]) and by Bell and Hincks ([4]) yielded values of $(0.44 \pm 0.10) \cdot 10^5$ s^{-1} and $(0.55 \pm 0.15) \cdot 10^5$ s^{-1} respectively. A weighted mean of all the estimates gives

$$\Lambda_i = (0.40 \pm 0.03) \cdot 10^5 \text{ s}^{-1} . \tag{5}$$

It remains to compare this result with a theoretical prediction. It is recalled that the simplest model, assuming a point nucleus, requires that the muon capture rate should have a Z^4 dependence. Wheeler took into account the finite size of the charge distribution and introduced an effective value of Z ([1]). For carbon, Z_{eff} is 5.75. Primakoff extended the treatment to include the limitations imposed on the final states by the Pauli exclusion principle ([2]). Finally, Flamand and Ford noted that the finite size of the nucleus modifies the muon wave function—an effect that reduces the muon density within the nucleus ([15]). The theory of Primakoff as refined by Fla-

([13]) R. A. Lundy, J. C. Sens, R. A. Swanson, V. L. Telegdi and D. D. Yovanovitch: *Phys. Rev. Lett.*, **1**, 102 (1958).

([14]) T. H. Fields, R. L. McIlwain and J. G. Fetkovich: *Bull. Am. Phys. Soc.*, **4**, 81 (1959).

([15]) G. Flamand and K. W. Ford: *Phys. Rev.*, **116**, 1591 (1959).

MAND and FORD, requires a total absorption rate for carbon of $0.41 \cdot 10^5\ s^{-1}$. The agreement with experiment is seen to be very good, though it should be borne in mind that the theory actually predicts only the trend in the variation of the capture rate with Z; it does not necessarily give the exact value for each individual nucleus.

Lastly, it should be noted that the small change in the total absorption rate from the value previously adopted does not significantly affect the recent measurements of the partial rate for the reaction leading to the ground state of ^{12}B (7).

* * *

The author wishes to express his gratitude to Dr. WILSON M. POWELL, Dr. ROBERT W. BIRGE, and the other members of the propane chamber group, without whose help this work, would not have been possible. He is indebted to the Bevatron crew and staff, and also to the data-reduction group led by HOWARD S. WHITE. It is also a pleasure to thank Dr. ROBERT D. SARD for useful discussions.

Note added in proof.

A measurement of the total absorption rate of muons in carbon has recently been reported by R. A. REITER, T. A. ROMANOWSKI, R. B. SUTTON and B. G. CHIDLEY (*Phys. Rev. Lett.*, **5**, 22, (1960)). Their value of $(0.373 \pm 0.011) \cdot 10^5\ s^{-1}$ is in excellent agreement with the result found in this work.

RIASSUNTO (*)

I muoni negativi arrestati in una camera a bolle a propano formano atomi mesici con i nuclei di carbonio. Successivamente o decadono nel loro modo usuale o interagiscono col nucleo. Sulla base di un campione composto di 2519 mesoni, si trova che la probabilità di interazione è $(7.4 \pm 0.8)\%$ ed il rapporto totale di assorbimento dei muoni nel carbonio diviene $(0.36 \pm 0.04) \cdot 10^5\ s^{-1}$. Si riscontra che il risultato è in buon accordo con la teoria.

(*) *Traduzione a cura della Redazione.*

Optical Spectra of Eu^{2+} and Gd^{3+} in CaF_2 (*).

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(ricevuto il 22 Aprile 1960)

This note reports spectra of Eu^{2+} and Gd^{3+} in a single crystal of CaF_2. The absorption lines observed in the case of Eu^{2+} indicate that there are transitions within the f^7 configuration. This seems to be the first time that transitions within the configuration have been definitely observed for this ion.

The ground state of both ions is $^8S_{\frac{7}{2}}$. Both ions can be substituted for calcium in the lattice of CaF_2. Paramagnetic resonance has shown that Eu^{2+} is situated at a site of cubic symmetry (1). Gd^{3+} can be situated either at a site of cubic or of axial symmetry (1-3).

1. - The spectrum of Gd^{3+}.

In a crystal containing less than 0.01% Gd sharp but weak lines were observed centered around 32 500 cm^{-1}. Additional very weak lines were observed at about 37 000 cm^{-1} These lines correspond to similar transitions observed by Dieke in $GdCl_3 \cdot 6H_2O$ (4,5). These have been interpreted as transitions to the ^6P and ^6I levels respectively.

2. - The spectrum of Eu^{2+}.

In a crystal to which about 0.1% Eu^{2+} was added to the melt, sharp absorption bands were found in the region 24 200, 32 300, 35 400 and 37 000 cm^{-1}. Some of these bands are shown in Fig. 1. In addition there are diffuse and strong bands centered around 29 000 cm^{-1} and 45 050 cm^{-1} with f number of the order of 0.01 and 0.03 respectively. The first band shows a peculiar staircase structure with peaks at 25 000, 26 000, 26 500,

(*) Supported in part by the Air Research and Development Command U.S. Air Force, through its European Office.

(1) C. Ryter: *Helv. Phys. Acta*, **30**, 358 (1957).

(2) J. M. Baker, B. Bleaney and W. Hayes: *Proc. Roy. Soc.*, A **247**, 141 (1958).

(3) W. Low: *Phys. Rev.*, **109**, 275 (1959).

(4) G. H. Dieke and L. A. Hall: *Journ. Chem. Phys.*, **27**, 465 (1957).

(5) G. H. Dieke and L. Leopold: *Journ. Opt. Soc. Am.*, **47**, 944 (1957).

(6) F. D. S. Butement: *Trans. Far. Soc.*, **44**, 617 (1948).

27 000, 28 300 and 29 700 cm^{-1} in ascending intensity. A spectrum of these diffuse bands at room temperature on a 2 mm thick sample is shown in Fig. 2.

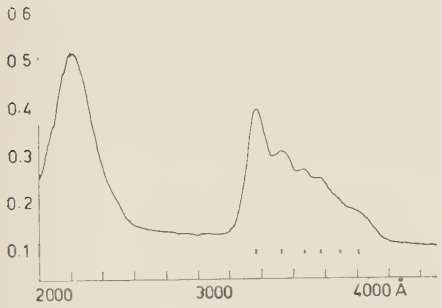

Fig. 1. – Spectrum of Eu^{2+} in CaF_2. The figure shows some of the weak and sharper absorption bands in the ultraviolet.

The spectrum of Eu^{2+} has been studied by a number of people both in solution and in solids. Butement (6) reports two broad bands of fairly strong intensity in $SrCl_2$ at 31 200 and 40 300 cm^{-1} with sharp lines superimposed on the first band. Katcoff and Freed (7) who studied the spectrum in $SrCl_2$ at low temperatures report strong bands in the region of (25 000÷÷34 000) cm^{-1} with superimposed vibrational structure. These bands have been interpreted by these authors to arise from transitions to a different configuration or even from an electron transfer spectrum.

The very weak and sharp lines observed by us seem to be indicative of transitions within the f^7 configuration similar to those observed for Gd^{3+}. Probably the first three groups correspond to transitions to ^6P, ^6I and ^6D levels. The lower nuclear charge in the case of Eu^{2+} results in scaling down of the Coulomb energy similar to that observed in the cases of the isoelectronic Sm^{2+} and Eu^{3+}.

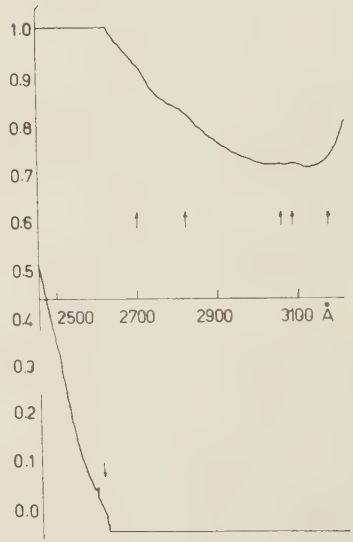

Fig. 2. – Spectrum of Eu^{2+} in CaF_2. The spectrum shows the strong and diffuse absorption bands.

The strong and diffuse bands are presumably transitions to the configuration $4f^6 5d$. This configuration yields the pentad ^{8}HGFDP. All these levels are closely spaced (8) and it is difficult to assign the observed bands.

(7) S. Katcoff and S. Freed: *Physica*, **14**, 17 (1948).

(8) W. Albertson: *Astrophys. Journ.*, **34**, 26 (1936).

IL NUOVO CIMENTO Vol. XVII, N. 4 16 Agosto 1960

On the Asymptotic and Causality Conditions in Quantum Field Theory - II.

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(ricevuto il 6 Giugno 1960)

Some conclusions reached in a recent paper [1] on the connection between the interacting field operator approach to quantum field theory as worked out by Lehmann, Symanzik and Zimmermann and the functional derivative approach to quantum field theory as proposed by Bogolubov are established in more details. Especially it is shown that Bogolubov's causality condition is a necessary integrability condition for the retarded or advanced solutions of the inhomogeneous Klein-Gordon equation. Associated with this the fact is investigated in some detail (related to Haag's theorem) that the transformation operator which connects the interacting field operator with the incoming or outgoing free field operator can only be unitary up to a positive renormalization constant smaller than one for real interactions. Its consequences for the commutation relations of the free field operators coinciding with the interacting field operator at an arbitrary time t are pointed out. Further the differences are discussed existing between both approaches with respect to the extrapolation of the reduced S-matrix elements off the mass shell (as considered in the theory of dispersion relations). Concluding some consequences are investigated which follow from the arbitrariness in the choice of the θ-functions of the retarded or advanced commutators in the S-matrix elements reduced in the interacting field operator approach (compare [1]). It is concluded that the reduction formulae derived by the use of the asymptotic condition for the interacting field operator can only be meaningful if the commutator condition for the interacting field operator is fulfilled, *i.e.* that the asymptotic condition and the commutator condition for the interacting field operator cannot be considered as separate conditions. It follows from these investigations that the consequences of a possible non-causal structure of quantum field theory can only be studied in the more general functional derivative approach to quantum field theory which avoids completely the introduction of an interacting field operator and the use of the corresponding asymptotic condition which are really quite unnecessary for the derivation of the reduction formulae for the S-matrix elements.

This paper will be published elsewhere.

[1] F. Kaschluhn: *Nuovo Cimento*, **12**, 541 (1959).

2817

IL NUOVO CIMENTO Vol. XVII, N. 4 16 Agosto 1960

Cosmic Ray Event of May 4, 1960.

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(ricevuto il 13 Giugno 1960)

An increase of the nucleonic component of cosmic rays was recorded at Prague station (geomagn. 50° 00′ N, 97° 13′ E 228 m a.s.l.) on May 4th, 1960 by means of a standard neutron monitor. The increase began at (10.33 ± 3) min UT and reached a maximum of $(11 \pm 3)\%$ (measured at five minutes intervals) at 10.50 UT. The normal level was reached by decay of the form $\sim t^{-\frac{3}{2}}$ during the next two hours. Neither increase in meson component nor any evident increase $((1.2 \pm 0.8)\%)$ in total intensity measured as double coincidences by standard cubical telescopes was observed. All values are related to the preoccurence level.

The same cosmic ray event was observed at the polar station Ellsworth (1), and at the mountain station (2 634 m a.s.l.) Lomnický štít (2), where the increase reached 110% and 14% respectively).

A revived and quickly changing spot-group (15° N, Central Meridian Passage April 27th) was on the west solar limb on May 4th, 1960. A system of loop prominences quickly developed in the coronal region above the sunspot group after 10 h UT. The evolution of the loop system and its accompanying terrestrial effects were observed at the Ondřejov Observatory. The most conspicuous event was the expansion of a bright loop at about 10.10 UT.

All conspicuous changes in the loop system were accompanied by radio events ($\lambda = 130$ cm, 56 cm, 37 cm). The enhancements of atmospherics (27 MHz), sudden phase anomaly and short wave fade-out in a broad frequency band also coincided in time with the evolution of the loop system (3). The intensity of all the accompanying phenomena equalled that of a flare with importance 3 or 3+.

No other H-alpha phenomenon interfered with the formation of loop prominences. The flare of importance 1+ ended at 9.12 and a small short-lived surge accompanying it ended before the onset of loop activity.

If we try to correlate the described cosmic ray event with some solar phenomenon, there seems to be only one possibility, namely the connection with the loop prominence.

(1) O. R. Santochi, J. R. Manzano, J. G. Roederer: preprint.

(2) P. Chaloupka, M. Jireš, T. Kowalski: preprint.

(3) J. Kleczek, L. Křivský: *Bull. Astr. Inst. Czechosl.*, **11**, 165 (1960).

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On the μ-e Mass Difference.

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(ricevuto il 12 Luglio 1960)

Lee and Yang have recently investigated various consequences of the existence of a vector boson which would transmit the weak interactions [1]. The equality of the β-decay and muon capture rates and also the experimental branching ratio value [2] $w_{\pi\to e+\nu}/w_{\pi\to\mu+\nu} \cong 10^{-4}$ seem to indicate that if such a vector boson exists $g^2_{\mu\nu}/4\pi = g^2_{e\nu}/4\pi = g^2_{np}/4\pi \approx 10^{-6}$, where $g_{\mu\nu}$, $g_{e\nu}$, g_{np} are the charged boson rationalized coupling constants to the muon, electron and nucleon respectively.

On the other hand the muon and the electron seem to have the same interaction with the electromagnetic field and therefore if they have also the same coupling with the vector boson it is difficult to see how one could explain the mass difference as arising from an interaction. One way to reconcile an asymmetry in the interaction with the above mentioned experimental facts seems at present, to assume that in the self-energy effect $g_{\mu\nu}$ and $g_{e\nu}$ are different and are the unrenormalized coupling constants of the boson with the bare muon and bare electron respectively. In the π-decay, μ-capture and β-decay, $g_{\mu\nu}$ and $g_{e\nu}$ are the renormalized coupling constants of the boson with the physical muon and physical electron respectively, and they are equal as a result of renormalization effects. We leave the theoretical justification of the above assumption for further investigation and report briefly on the self-energy calculation results.

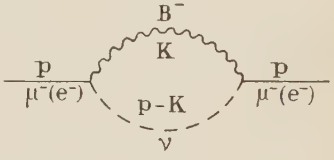

Fig. 1.

We assume that the muon and the electron have the same mass with respect to all interactions except the one with the charged boson.

The lowest order contribution to the self-energy due to this interaction is given by the matrix element of the Feynman diagram of Fig. 1. Remembering that the

[1] T.D. Lee and C. N. Yang: *Implications of the Intermediate Boson Basis of the Weak Interactions* (preprint).

[2] J. Ashkin, T. Fazzini, G. Fidecaro, A. W. Merrison, H. Paul and A. V. Tollestrup: *Nuovo Cimento*, **13**, 1240 (1959); H. L. Anderson, T. Fujii, R. H. Miller and L. Tau: *Branching Ratio of the Electronic Mode of Positive Pion Decay*, EFINS-60-15, Report No. 341 (submitted to *Phys. Rev.*).

vector boson propagator in momentum space is given by

$$\frac{\delta_{\nu\mu} - (1/m_B^2)k_\mu k_\nu}{m_B^2 + k^2}$$

and choosing the cut-off at the nucleon mass M this matrix element is

$$S^{(2)} = -\frac{ig_{\mu\nu}^2}{2\pi^2}\left(I_a - \frac{m}{2}I_b + \frac{3}{2}I_c + \frac{4\pi}{3}\frac{M^2}{m_B^2}\right),$$

with

$$I_a = \int \frac{\mathrm{d}^3k}{(E_k - m)^2 - k^2} \cong 2\pi M\left[\frac{M^2}{3\lambda m} + \frac{m_B}{64m}\left(\frac{5m_B^3}{M^3} + \frac{m_B^5}{M^5}\right)\right],$$

$$I_b = \int \frac{\mathrm{d}^3k}{E_k[(E_k - m)^2 - k^2]} \cong 2\pi M\,\frac{M}{2\lambda m},$$

$$I_c = \int \frac{\mathrm{d}^3k}{(m + k)^2 - E_k^2} \cong -2\pi M\,\frac{M^2}{2\lambda m},$$

where

$$E_k = \sqrt{m_B^2 + k^2}\,; \qquad \lambda = \frac{m_B^2}{2m}\,;$$

$m \to$ mass of the electron,

$m_B \to$ mass of the boson.

If $m_B/M \approx \frac{1}{2}$ we find then for the self-energy $\delta m = iS^{(2)}$ the result

$$\frac{\delta m}{M} \cong \frac{g_{\mu\nu}^2}{4\pi}\,0.4\cdot 10^2.$$

A value of $g_{\mu\nu}^2/4\pi \approx 3\cdot 10^{-3}$ gives therefore a self-energy contribution to the muon mass of the order of 10^8 MeV. It follows that we must choose $g_{e\nu}^2/4\pi < 1.5\cdot 10^{-5}$ in order to obtain a self-energy contribution to the electron mass smaller than m.

The experimental values of the muon and the electron gyromagnetic factors are [3,4]

$$g_\mu = 2(1.0015 \pm .0006)\,; \qquad g_e = 2(1.001165 \pm .000011).$$

An estimate of the lowest order contribution to g_μ and g_e of the neutrino charged lepton-charged boson interaction gives with the unrenormalized $g_{\mu\nu}$ and $g_{e\nu}$ an upper limit of 10^{-4} in the case of the muon and 10^{-8} in the case of the electron

([3]) P. Franken and S. Liebes: *Phys. Rev.*, **104**, 1197 (1956).
([4]) R. A. Lundy: *Phys. Rev. Lett.*, **1**, 38 (1958).

and with the renormalized

$$\frac{g_{\mu\nu}^2}{4\pi} = \frac{g_{e\nu}^2}{4\pi} \simeq 10^{-6},$$

the upper limits are 10^{-7} and 10^{-9} respectively.

* * *

I wish to thank Prof. G. Wataghin for many helpful discussions and encouragement throughout this work. I am also grateful to Dr. J. N. Chahoud for valuable discussions.

Operation at the «Medicina Station» of a 327 MHz Radiotelescope.

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(ricevuto il 13 Luglio 1960)

On June 9th 1960 came into operation near Medicina (*) a meridian transit radiotelescope of the Cambridge type (1) working at frequency of about at 327.4 MHz.

The instrument, shown in Fig. 1, is constitued by a parabolic cylinder mirror 110 m long and 6.7 m wide, oriented along the EW direction and which can be rotated around its axis so as to cover declinations from $-15°$ to $+90°$.

The surface of the cylinder is formed by 0.7 mm steel wires, spaced 3 cm and supported by ten parabolic sections. The wires are pulled at one side by springs; the tension of the wires and the mechanical tolerances of the supporting sections are such that the mirror never differs by more than 4 cm from the ideal surface. The supporting sections and hence the mirror are made to rotate simultaneously by motors.

On the focal line of the cylinder are placed 112 full wave dipoles, arranged in 16 groups of 7 each. The dipoles of each group are placed in parallel and feed, through a two-wire line, a flexible cable. The cables bring the signal down to a network of rigid copper coaxial cables air and teflon spaced, connected to the receiver. This network is such that the electrical lengths from each group of dipoles to the receiver are equal.

The power collected from the 16 groups of dipoles follows a Dolph-Tchebyscheff distribution for -30 db sidelobes level. This is accomplished by means of impedance matchers at the junctions of the coaxial cable network.

The receiver, of a conventional type but very carefully designed for stability, has a bandwidth of 2 MHz. The r.f amplifier employs as first stage a grounded grid P.T.T. 141 and the overall noise figure is 8 db. Calibrations are obtained by connecting the receiver to a saturated diode noise source.

The aerial is designed for a half-power points beam of $32' \times 7°$ and for a sensitivity of few units $\cdot 10^{-25}$ W/m^2 Hz with the present receiver. Preliminary

(*) The coordinates of the station are: $\delta = 44° 30' 26''$; $\varphi = -0^h 46^m 39^s$.

(1) M. Ryle: *Nature*, **180**, 110 (1957).

Fig. 1.

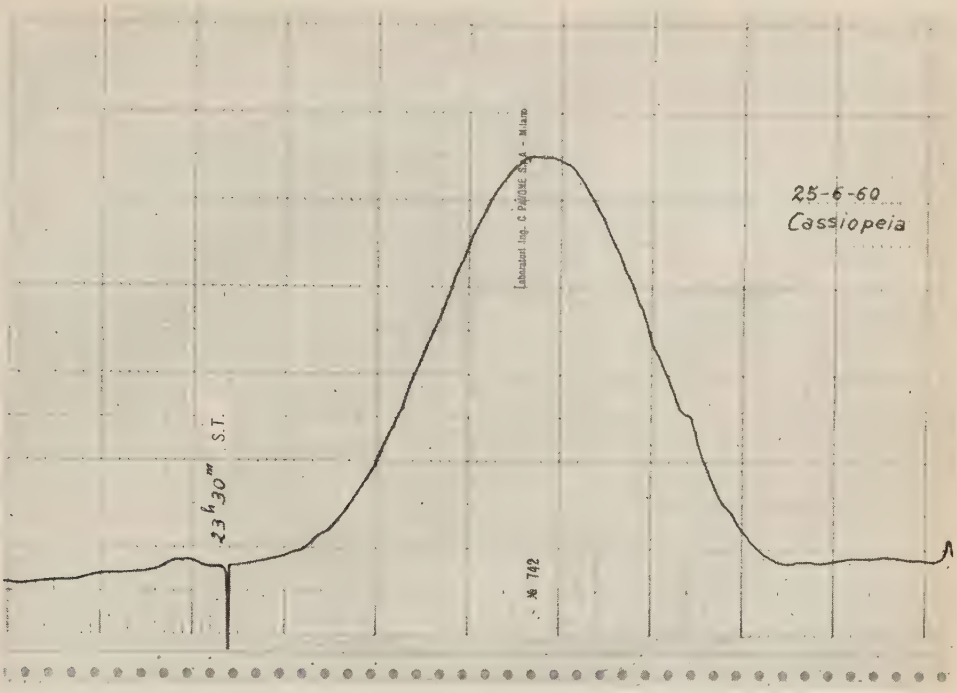

Fig. 2. – A transit record of the « Cassiopeia A » radiosource.

measurements have indicated that these requirements are essentially fulfilled.

The radiotelescope described here though being itself a useful tool for research, is primarily intended as a pilot for the construction of an instrument working at the same frequency which should have a collecting area about 30 times larger.

* * *

It is for us a pleasure to thank the Ministry of Public Education of Italy both for the financial support and for the interest taken in the project. Prof. G. Puppi is cordially acknowledged for his many advices and for enthousiastic support. Profs. M. Ryle, G. Righini, E. De Castro and Drs. D. Mathewson, M. Cichetti and G. Vannucchi as well as Mr. Piattelli are also acknowledged for the many useful suggestions.

We would like finally to thank the technicians of our group Messrs. Baldeschi, Cavallo, Fusi-Pecci, Galazzi, Righetti and Volta for the skilful construction of the instrument.

On the Indefinite Metric in the Lee Model (*).

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(ricevuto il 13 Luglio 1960)

Källén and Pauli (1) have devised a consistent set of rules for handling the Lee model when the coupling constant is pure imaginary. We want to point out that the metric operator that is needed in such a formalism is not unique and that it may also result in giving complex norms to states that may or may not contain the ghost state. Nevertheless, this freedom will be shown not to affect the interpretation of scattering experiments.

Let the probability of occupation of a state $\Psi(t)$ be $\langle \Psi(t) | \eta | \Psi(t) \rangle$. If $\Psi(t)$ satisfies the Schrödinger equation then

$$\eta H = H^{+} \eta \tag{1}$$

in order that the probability of occupation be independent of time. Let

$$H = H_0 + H_I$$

where in this case

$$H_0^{+} = H_0 , \qquad H_I^{+} = - H_I .$$

It is readily shown if

$$\eta = \exp \left[- i\varphi_1 \psi_V^{+} \psi_V - i\varphi_2 \psi_N^{+} \psi_N - i\varphi_3 \sum_k a^{+}(k) a(k) \right. , \tag{2}$$

with $\varphi_1 - \varphi_2 - \varphi_3 = (2n+1)\pi$, that eq. (1) is satisfied. Let the energy of the V-like particles (including the ghost state) be given by E. Then

$$\frac{g_0^2}{V} \sum \frac{[f(\omega)]^2}{2\omega(E - m_N - \omega)} = E - m_V^0 . \tag{3}$$

(*) Supported in part by the Atomic Energy Commission under contract no. AT(30-1)-2399 and by the National Science Foundation under contract no. NSF-G7062.

(1) G. Källén and W. Pauli: *Dan. Nat. Fys. Medd.*, **30**, No. 7 (1955).

If E_1 is largest root of eq. (3) which is less than $(m_N+\mu)$, then it follows that there will be an additional root smaller than E_1 if

$$1+\frac{g_0^2}{V}\sum\frac{[f(\omega)]^2}{2\omega(E_1-m_N-\omega)^2}<0\,,$$

and if

$$1+\frac{g_0^2}{V}\sum\frac{[f(\omega)]^2}{2\omega(E_1-m_N-\omega)^2}>0\,,$$

there will be no other root smaller than E_1. If E_2 is the second root of eq. (3) it follows that

$$1+\frac{g_0^2}{V}\sum\frac{[f(\omega)]^2}{2\omega(E_2-m_N-\omega)^2}>0\,.$$

Using the metric operator given above it follows that for these states

$$\langle V_{E_1,E_2}|\eta|V_{E_1,E_2}\rangle=\exp[-i\varphi_1]|c_{1,2}|^2\left[1-\frac{|g_0|^2}{V}\sum\frac{[f(\omega)]^2}{2\omega(E_{1,2}-m_N-\omega)^2}\right].$$

The metric operator of reference [1] is obtained with $\varphi_1=\pi$, $\varphi_2=\varphi_3=0$. With that choice, the states of eigenvalue E_1, E_2 have positive and negative norms respectively. And if there should be only one root to eq. (3) $(E<m_N+\mu)$ then the norm of this state is negative. But in general all norms will be complex if φ_1, φ_2, φ_3 are different from π. And the assignment of negative norm to the ghost state (root E_2) is a matter of choice.

If the single particle and scattering states of H should form a complete set, then it follows that

$$\eta^+=\sum_{\text{all states}}\eta^+|\varphi_i\rangle\langle\varphi_i|\eta|\varphi_i\rangle\langle\varphi_i|\eta^+\,. \tag{4}$$

From eq. (4) it follows that

$$S\eta^+S^+\eta=1\,. \tag{5}$$

where

$$\eta_{ab}=\langle\psi_a^{(\pm)}|\eta|\psi_b^{(\pm)}\rangle=\delta_{ab}\exp[i\varphi_a]\,,$$
$$S_{ab}=\langle\psi_a^{(-)}|\eta|\psi_b^{(+)}\rangle\,,$$
$$S_{ab}^+=S_{ba}^*\,,$$

and $\psi^{(\pm)}$ are scattering states. Applying eq. (5) to V, θ scattering when eq. (3) has two roots and using eq. (2) it follows that

$$\eta_{V\theta,V\theta}=-\exp[-i\varphi_1-i\varphi_3]\,,$$
$$\eta_{V_G\theta,V_G\theta}=\exp[-i\varphi_1-i\varphi_3]\,,$$
$$\eta_{N\theta'\theta'',N\theta'\theta''}=\exp[-i\varphi_2-2i\varphi_3]\,,$$
$$\sum_{\theta'}|S_{V\theta',V\theta}|^2+\sum_{\theta',\theta''}|S_{N\theta'\theta'',V\theta}|^2-\sum_{\theta'}|S_{V_G\theta',V\theta}|^2=1\,,$$

where

$$H|V\rangle = E_1|V\rangle ,$$

$$H|V_G\rangle = E_2|V_G\rangle .$$

And if eq. (3) should have only one root

$$\sum_{\theta'} |S_{V\theta', V\theta}|^2 - \sum_{\theta'\theta''} |S_{N\theta'\theta'', V\theta}|^2 = 1 .$$

Thus the assignment of negative probabilities to different transitions is independent of the choice of the norms. Whether this result will hold in general for any Hamiltonian with a complex coupling constant is not clear.

Subtractions in Forward Scattering Dispersion Relations (*).

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The question has been raised recently (1) whether all physical amplitudes may not satisfy dispersion relations which do not require any subtractions. In this Letter we would like to point out first of all that this possibility may be excluded empirically for the pion-nucleon forward scattering amplitude, on the basis of the present experimental data.

The unsubtracted dispersion relations for the forward scattering amplitudes for charged pions scattered by protons may be written (2)

$$D_+(E) = -\frac{2f^2}{E - \frac{1}{2}M} + \frac{1}{4\pi^2}\int_1^\infty \left[\frac{\sigma_+(E')}{E' - E} + \frac{\sigma_-(E')}{E' + E}\right] p'\,\mathrm{d}E' \,, \tag{1a}$$

$$D_-(E) = +\frac{2f^2}{E + \frac{1}{2}M} + \frac{1}{4\pi^2}\int_1^\infty \left[\frac{\sigma_-(E')}{E' + E} + \frac{\sigma_+(E')}{E' + E}\right] p'\,\mathrm{d}E' \,, \tag{1b}$$

where $D_\pm(E)$ denote the real parts of the $\pi^\pm$-p forward scattering amplitudes at incident pion energy E in the laboratory system, M is the proton mass, and f^2 is the renormalized pion-nucleon coupling constant. Evaluating these at zero pion kinetic energy and adding, we obtain the following expression

$$D_+(1) + D_-(1) = -\frac{2f^2}{M(1 - \frac{1}{4}M^2)} + \frac{1}{2\pi^2}\int_0^\infty [\sigma_+(p) + \sigma_-(p)]\,\mathrm{d}p \,. \tag{2}$$

(*) Work carried out under the auspices of the U.S. Atomic Energy Commission.

(1) S. Drell: *Proceedings of the 1958 Annual International Conference on High Energy Physics at CERN* (edited by B. Ferretti, Geneva, 1958), p. 32; S. Drell and F. Zachariasen: *Phys. Rev.*, **111**, 1727 (1958). M. Sugawara (preprint received May, 1960) has recently suggested that the forward pion-nucleon scattering dispersion relations may also be valid without subtractions.

(2) We choose units such that $\hbar$, c and the pion mass each has magnitude unity.

in which p denotes the laboratory momentum of the pion. The left-hand side can be expressed in terms of the zero-energy scattering lengths, a_1 and a_3 for the $T = \frac{1}{2}$ and $\frac{3}{2}$ pion-nucleon states. Using the values of a_1 and a_3 given by OREAR [3], $[D_+(1)+D_-(1)]$ takes the value

$$D_+(1) + D_-(1) = \frac{2}{3}\left(1 + \frac{1}{M}\right)(a_1 + 2a_3) = -0.046 \,. \tag{3}$$

The more recent values $a_1 = 0.178$, $a_3 = -0.087$ given by HAMILTON and WOOLCOCK [4] lead to $+0.003 \pm 0.008$ for this sum. The first term on the right-hand side of eq. (2) is quite well known, its value being -0.024 for $f^2 = 0.08$; the present evidence [5] is that this value is correct to better than 10%. The remaining integral on the right of eq. (2) is necessarily a positive quantity and its value is large relative to the first term. The cross-section σ_+ is now known [6] up to $p = 4$ GeV/c; σ_- has been measured [6] at 4.3 GeV/c and 5.2 GeV/c, but otherwise the experimental data [5,7] extend only to 1.8 GeV/c. Integrating up to 1.8 GeV/c already leads to a value 2.48. Although it is difficult to estimate accurately the limits of error on this value, it is easy to obtain a lower limit; a gross underestimate of all the measured cross-sections still leads to a value of 1.5. From $p = 1.8$ GeV/c to $p = 4$ GeV/c, the sum $(\sigma_+ + \sigma_-)$ has a value of about 50 mb wherever it is known, so that a further contribution of order 2.0 is to be expected from this region. Whatever contribution comes from beyond 4.0 GeV/c must be positive, so that the right-hand side has at least a value of about 3.5. The relation (2) therefore cannot be valid [8] regardless of the behavior of the total cross-sections at higher energies.

Recently AMATI, FIERZ and GLASER [9] have pointed out that the assumption that the forward scattering amplitudes $F_\pm(E) = D_\pm(E) + iA_\pm(E)$ have a high energy behaviour such that

$$\lim_{E\to\infty} \frac{F_\pm(E)}{E} = \text{constant} \tag{4}$$

allows a demonstration of the Pomeranchuk theorem that $\lim_{E\to\infty} [\sigma_+(E) - \sigma_-(E)] = 0$, this limit being such that the integral

$$\int_1^\infty \frac{\sigma_-(E')}{p'} \, \sigma_+(E') \, \mathrm{d}E' \tag{5}$$

(3) J. OREAR: *Phys. Rev.*, **96**, 176 (1954).

(4) J. HAMILTON and W. S. WOOLCOCK: *Phys. Rev.*, **118**, 291 (1960).

(5) H. P. NOYES and D. N. EDWARDS (UCRL report No. 5771-T, November, 1959) have given the most complete evaluation of f^2 from the pion-nucleon scattering data to date.

(6) M. J. LONGO, J. A. HELLAND, W. N. HESS, B. J. MOYER and V. PEREZ-MENDEZ: *Phys. Rev. Lett.*, **3**, 568 (1959) and the references given therein.

(7) T. J. DEVLIN, B. C. BARISH, W. N. HESS, V. PEREZ-MENDEZ and J. SOLOMON: *Phys. Rev. Lett.*, **4**, 242 (1960).

(8) G. FELDMAN and R. ARNOWITT (*Phys. Rev.*, **108**, 144 (1957)) have given an argument that at least one subtraction must be made for the forward-scattering pion-nucleon dispersion relations if the pion-pion coupling constant (the coefficient λ of a $\lambda\varphi^4$ interaction in the Hamiltonian) is different from zero.

(9) D. AMATI, M. FIERZ and V. GLASER: *Phys. Rev. Lett.*, **4**, 89 (1960), hereafter referred to as AFG.

is convergent. This condition (4) is sufficient to guarantee the validity of the twice-subtracted dispersion relations

$$(6a) \qquad D_+(E) = \frac{1}{2}(1+E)\,D_+(1) + \frac{1}{2}(1-E)\,D_-(1) + \\ + \frac{2f^2p^2}{E-1/2M} + \frac{p^2}{4\pi^2}\int_1^\infty \left[\frac{\sigma_-(E')}{E'-E} + \frac{\sigma_+(E')}{E'+E}\right]\frac{\mathrm{d}E'}{p'},$$

$$(6b) \qquad D_-(E) = \frac{1}{2}(1+E)\,D_-(1) + \frac{1}{2}(1-E)\,D_+(1) - \\ - \frac{2f^2p^2}{E+1/2M} + \frac{p^2}{4\pi^2}\int_1^\infty \left[\frac{\sigma_-(E')}{E'-E} + \frac{\sigma_+(E')}{E'+E}\right]\frac{\mathrm{d}E'}{p'},$$

as may be shown by considering the Cauchy integral for $F(E')/(E'^2-1)(E'-E)$ around a suitable contour.

AFG also argued that the convergence of (5) justified the use of an unsubtracted dispersion relation for $[D_+(E) - D_-(E)]$. However, on the basis of their assumption (4), this conclusion does not follow. In fact from (6*a*) and (6*b*), we have

$$(7) \qquad D_+(E) - D_-(E) = D_+(1) - D_-(1)E + \frac{2Ep^2}{4\pi^2}\int_1^\infty \frac{\sigma_+(E') - \sigma_-(E')}{(E'^2-E^2)p'}\,\mathrm{d}E' + \frac{4f^2p^2E}{E^2-1/4M^2}.$$

In virtue of the convergence of (5), this relation may be rewritten

$$(8) \qquad \frac{D_+(E) - D_-(E)}{E} + \frac{4f^2(1-1/4M^2)}{E^2-1/4M^2} - \frac{1}{2\pi^2}\int_1^\infty \frac{[\sigma_+(E') - \sigma_-(E')]p'\,\mathrm{d}E'}{E'^2-E^2} = \\ = D_+(1) - D_-(1) + 4f^2 - \frac{1}{2\pi^2}\int_1^\infty \frac{\sigma_+(E') - \sigma_-(E')}{p'}\,\mathrm{d}E'.$$

As AFG showed, the left-hand side approaches

$$(9) \qquad \lim_{E\to\infty} \frac{D_+(E) - D_-(E)}{E}$$

as E approaches infinity. This limit (9) is therefore not in general zero, but takes the value given by the right-hand side of (8), a value not yet known from any theoretical argument.

It has frequently been speculated ([10,11]) that the right-hand side of (8) is indeed

([10]) M. L. Goldberger, H. Miyazawa and R. Oehme: *Phys. Rev.*, **99**, 986 (1955).
([11]) T. D. Spearman: to be published.

zero. In fact, this is not in contradiction with the experimental data available to date on pion-nucleon scattering, as shown by the close agreement of the coupling constant f^2 deduced by SPEARMAN ([11]) assuming this to be the case and the values obtained from other analyses ([5,12]) not dependent on this assumption. However, this result has not been proved, and remains a challenging speculation.

* * *

In conclusion the author wishes to thank Professors R. H. DALITZ and R. OEHME for suggesting this work and for helpful discussions.

([12]) G. SALZMAN and H. SCHNITZER: *Phys. Rev.*, **112**, 1802 (1958).

LIBRI RICEVUTI E RECENSIONI

W. T. Scott – *The Physics of Electricity and Magnetism*. J. Wiley & S., Inc., 1959, pp. 635.

Questo libro è il risultato delle lezioni svolte dal prof. Scott allo Smith College. La materia è trattata al livello di un corso di fisica per fisici, ingegneri e chimici, quale viene ad es. svolto nel II anno della nostra Facoltà di Scienze.

Il libro ha il pregio di costituire una chiara esposizione dei concetti classici e nello stesso tempo di contenere risultati recenti e argomenti di attualità e di porre un continuo accento sulla struttura atomica dei corpi, specialmente in vista di un corso successivo di fisica dello stato solido.

Come promesso dallo stesso Scott nella prefazione, teoria ed esperienza, sviluppi matematici e concetti fisici si bilanciano opportunamente, favorendo al massimo l'intuito fisico del lettore. Numerosi esercizi a vario livello di difficoltà sono inseriti nel testo. Dopo un primo capitolo sui fondamenti della elettrostatica in assenza di materia, due capitoli sono dedicati rispettivamente ai conduttori e ai dialettrici con una breve introduzione aggiornata sulle caratteristiche fondamentali dei metalli e degli isolanti. Il cap. V tratta le correnti stazionarie nel mondo convenzionale. Il cap. VI è intitolato molto felicemente ai fenomeni elettrici connessi con i livelli energetici degli elettroni. I due capitoli successivi trattano il magnetismo nel vuoto e nella materia insistendo sulla sua connessione con le correnti atomiche; il capitolo IX è dedicato alle correnti alternate e l'ultimo capitolo alla propagazione delle onde elettromagnetiche. È dimostrata anche la invarianza delle equazioni di Maxwell alle trasformazioni di Lorentz, come breve introduzione alla teoria della relatività. Le questioni puramente matematiche sono esposte alla fine del libro in quattro appendici molto dettagliate.

Ritengo che questo nuovo trattato di elettromagnetismo possa risultare molto utile per studenti interessati ad approfondire gli argomenti trattati nei testi di studio (per es. come complemento alle dispense di Fisica Sperimentale del prof. Amaldi); per quanto a livello scolastico, il trattato susciterà certamente interesse anche nel docente e nel ricercatore.

Giovanni Boato

L. Brillouin – *La Science et la Theorie de l'Information*. Ed. Masson & Cie. Paris, 1959, pp. 302, Fr. 4800.

Dopo l'edizione in lingua inglese (Academic Pr. Inc., 1956) che già ha reso assai nota e meritatamente questa opera, appare la traduzione in lingua francese eseguita da M. Parodi. Le numerose correzioni e le aggiunte di complementi a qualche capitolo fatte dall'autore (cito per es. il breve paragrafo sul problema degli integrali divergenti in meccanica quantica al cap. 16) giustificano almeno in parte la comparsa

di questa edizione in francese. È infatti assai dubbio che la precedente edizione costituisse una difficoltà linguistica per il pubblico a cui il libro è indirizzato.

Nei 20 capitoli in cui il libro è diviso dopo la definizione e la discussione del concetto di informazione (Cap. I, II), vengono discussi gli aspetti del linguaggio, i principi generali e l'uso delle espressioni in codice, il problema degli errori ad essi connessi e sono date applicazioni dei metodi esposti (Cap. III-VII). Il cap. VIII è dedicato all'analisi dei segnali con la serie di Fourier. Nei capitoli seguenti (IX-XVI) vengono discussi diversi problemi nati dalla fisica: i principi della termodinamica, il moto browniano, disturbi termici nei circuiti elettrici, la relazione fra informazione ed entropia, il demonietto di Marxwell, le relazioni di indeterminazione.

È dopo la lettura attenta di questi capitoli che si può fare un bilancio circa il contributo che la teoria delle informazioni porta alla nostra conoscenza: essa è certamente utile nelle discussioni di principio poichè con le sue classificazioni e i suoi metodi migliora l'economia della discussione stessa, ma essa costituisce ancora uno strumento assai incompleto quando la si vuole usare per predire fatti o per applicazioni pratiche.

Dopo un'ulteriore indagine sulle informazioni nelle telecomunicazioni, nelle operazioni di scrittura e lettura, nel problema delle macchine calcolatrici (cap. XVII-XIX) questi rilievi vengono suggeriti dall'autore, mitigati dall'ottimismo nei futuri sviluppi (cap. XX).

L'incompletezza della teoria viene attribuita al fatto che ad essa è estranea una qualsiasi valutazione dell'importanza relativa delle informazioni alle quali viene dato in ogni caso lo stesso peso. Il problema del valore delle informazioni è ancora aperto e il libro si conclude coi tentativi fatti mediante gli strumenti della logica simbolica per impostarne la trattazione. La veste tipografica è ben curata ma inferiore a quella della edizione in inglese. Il prezzo del libro purtroppo non segue lo stesso tipo di variazione.

V. NARDI

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